Discrete Convexity for Multiflows and 0-extensions

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Abstract: This paper addresses an approach to extend the submodularity and L-convexity concepts to more general structures than integer lattices. The main motivations are to give a solution of the tractability classification in the minimum 0-extension problem and to give a discrete-convex-analysis view to combinatorial multiflow dualities.

Keywords: Submodular functions, L-convex functions, modular lattices, modular graphs, multicommodity flows, 0-extensions, valued CSP, fractional polymorphism

1 Introduction

This paper addresses an approach, developed in [12], to extend the submodularity and L-convexity concepts to more general structures than integer lattices. The main motivations of this approach are to give a solution of the tractability classification in the minimum 0-extension problem [19, 21] and to give a discrete-convex-analysis view to combinatorial multiflow dualities [8, 9, 10, 11, 20].

The central of this approach is the concept of L-convex functions on certain oriented graphs which have many properties analogous to that L$^\natural$-convex functions have. An *L$^\natural$-convex function* [27] is a function g on $\mathbf{Z}^n$ satisfying

$$g(p) + g(q) \geq g(\lfloor (p+q)/2 \rfloor) + g(\lceil (p+q)/2 \rceil) \quad (p, q \in \mathbf{Z}^n), \tag{1.1}$$

where $\lfloor \cdot \rfloor$ (resp. $\lceil \cdot \rceil$) is an operation on $\mathbf{R}^n$ that rounds down (resp. up) the decimal fraction of each components. We start with regarding L$^\natural$-convex functions as functions on a graph. Observe that any function on $\mathbf{Z}^n$ is identified with a function on the vertex set of an n-dimensional grid, where an *n-dimensional grid* (an *n-grid* for short) Gr_n is a graph isomorphic to the graph on $\mathbf{Z}^n$ obtained by joining an edge for each pair of $p, q \in \mathbf{Z}^n$ with $\|p - q\|_1 = 1$. The vector ordering $\leq$ on $\mathbf{Z}^n$ induces the orientation of the n-grid so that each edge pq is oriented as $p \leftarrow q$ if $p \leq q$. An orientation o obtained in this way is called a *linear orientation.* Then one can observe that an L$^\natural$-convex function is well-defined as a function on the vertex set on the linearly-oriented n-grid, and many of the intriguing properties can be described as graphical terms.

This observation gives rise to a question: *Is there any graph, generalizing the n-grid, that deserves to be the domain of what we should call L-convex (L$^\natural$-convex) functions ?* The study of [12] suggests that an *orientable modular graph* is such a graph. An undirected graph Γ is *modular* if every triple of vertices has a median, where a *median* of vertices u_1, u_2, u_3 is a vertex m satisfying

$$d_\Gamma(u_i, u_j) = d_\Gamma(u_i, p) + d_\Gamma(p, u_j) \quad (1 \leq i < j \leq 3).$$

An undirected graph Γ is *orientable* if there is an orientation such that, for every 4-cycle $uu', u'v', vv', uv$, uu' is oriented as $u \to u'$ if vv' is oriented as $v \to v'$. An orientation o of this property is called *admissible.*

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Obviously an n-grid is modular, and an linear orientation is admissible. A pair (Γ, o) of an orientable modular graph Γ and an admissible orientation o is called a *modular complex*. Orientable modular graphs appeared Karzanov's works on the minimum 0-extension problems [19, 21]. It turned out in [12] that a modular complex is a reasonable generalization of a linearly-ordered n-grid, and has a sufficiently rich structure that enables us to develop a theory of L-convex functions on it.

In Section 2, we explain a theory of *submodular functions on modular semilattices*, which plays a role analogous to that of submodular functions on distributive lattices in discrete convex analysis. In Section 3, we explain a theory of *L-convex functions on modular complexes*.

In Section 4, we explain how modular complexes and L-convex functions arise from minimum 0-extension problems and multicommodity flow problems, and explain how to apply our theory to these problems.

This paper is regarded as an extended abstract of [12, 13, 14]. See these papers for the details.

Notation. $\mathbf{R} \cup \{+\infty\}$ and $\mathbf{Q} \cup \{+\infty\}$ are denoted by $\bar{\mathbf{R}}$ and $\bar{\mathbf{Q}}$. Let $\mathcal{L}$ be a partially ordered set $\mathcal{L}$ with partial order $\preceq$. For $p, q \in \mathcal{L}$, the highest common lower bound of p, q, if it exists, is denoted by $p \wedge q$, and the least common upper bound of p, q, if it exists, is denoted by $p \vee q$. A pair $p, q \in \mathcal{L}$ is said be *bounded* if p, q have a common upper bound. $\mathcal{L}$ is called a *(meet-)semilattice* if $p \wedge q$ exists for every $p, q \in \mathcal{L}$, and $\mathcal{L}$ is called a *lattice* if both $p \wedge q$ and $p \vee q$ exist for every $p, q \in \mathcal{L}$. It is easy to see that if $\mathcal{L}$ is a meet-semilattice and a pair $p, q \in \mathcal{L}$ is bounded, then $p \vee q$ exists. The lowest element is denoted by $\mathbf{0}$, and the highest element is denoted by $\mathbf{1}$ if exists.

The rank $r(p)$ of an element p is the maximum length of a chain between $\mathbf{0}$ and p. A lattice $\mathcal{L}$ is called *modular* if r satisfies $r(p) + r(q) = r(p \wedge q) + r(p \vee q)$ for $p, q \in \mathcal{L}$. For $p \preceq q$, the interval $[p, q]$ is the set of elements p' with $p \preceq p' \preceq q$, and $r[p, q]$ is the maximum length of a chain between p and q. If $\mathcal{L}$ is modular, then $r[p, q] = r(q) - r(p)$.

2 Submodular functions on modular semilattices

In this section, we introduce *submodular functions on modular semilattices* as building blocks of our theory. A modular semilattice is a semilattice analogue of a modular lattice, which was introduced by Bandelt, Van de Vel, and Verheul [1].

A modular semilattice $\mathcal{L}$ is not necessarily a lattice. Join $p \vee q$ of elements p, q may not exist. However we can define a kind of a join, called a *fractional join*, which is a formal convex combination of a set $\mathcal{E}^{p,q}$ of elements determined by (p, q):

$$\sum \{\nu_u u \mid u \in \mathcal{E}^{p,q}\}.$$

If p, q have the join $p \vee q$, then the fractional join is equal to $1(p \vee q)$. In Section 2.1, we explain how to define set $\mathcal{E}^{p,q}$ and its coefficient ν. Then we can naturally define a *submodular function* on $\mathcal{L}$ by a function $f : \mathcal{L} \to \bar{\mathbf{R}}$ satisfying

$$f(p) + f(q) \geq f(p \wedge q) + \sum_{u \in \mathcal{E}^{p,q}} \nu_u f(u) \quad (p, q \in \mathcal{L}).$$

In Section 2.2, following this idea, we introduce submodular functions on modular semilattices. We do not know whether our submodular function f can be efficiently minimized, as in the case of submodular functions on distributive lattices [7, 18, 29]. However, if f is given as a special form, *a sum of submodular functions of bounded arity*, then f can be efficiently minimized. This is an application of a recent breakthrough of Thapper and Živný [30] on *Valued Constraint Satisfaction Problem* (VCSP). In Section 2.3, we explain the tractability of the submodular function minimization under VCSP model. The class of modular semilattices is broad, and hence the class of our submodular functions is broad. Special cases include bisubmodular functions (see [5, Section 3.5]), k-submodular functions [16], and α-bisubmodular functions [15]. We explain such examples in Section 2.4.

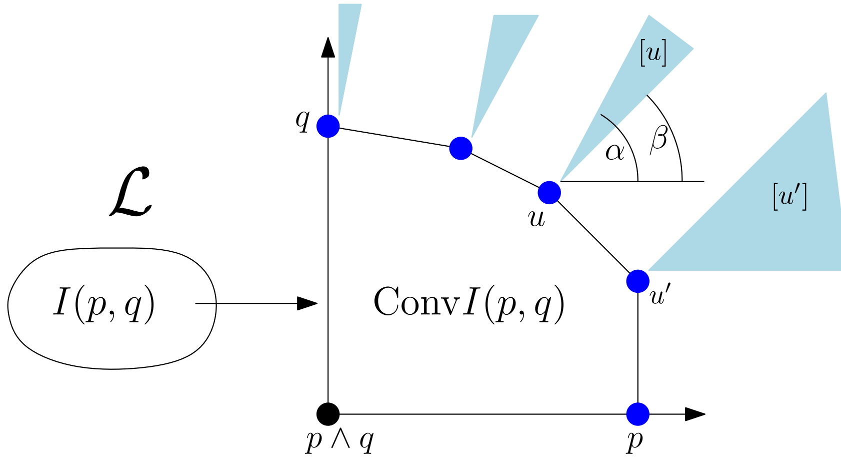

Figure 1: (p,q)-envelope

2.1 Modular semilattices

Definition 2.1 ([1])**.** A semilattice $\mathcal{L}$ is called *modular* if $[\mathbf{0}, p]$ is a modular lattice for every $p \in \mathcal{L}$, and every pairwise bounded triple u, v, w has a common least upper bound $u \vee v \vee w$.

Let $\mathcal{L}$ be a modular semilattice. As mentioned already, there is an appropriate notion of the *fractional join*, which is a formal convex combination $\sum\{\nu_u u \mid u \in \mathcal{E}^{p,q}\}$ of the set $\mathcal{E}^{p,q}$ of elements, called the (p,q)-*envelope*, determined by each pair (p,q). See Figure 1. As in Figure 1, (p,q)-envelope $\mathcal{E}^{p,q}$ is associated with the set of maximal extreme points in a polygon $\mathrm{Conv}\, I(p,q)$ in $\mathbf{R}^2_+$. The coefficient ν_u is given by the valuation ν of the normal cone $[u]$ of u in $\mathrm{Conv}\, I(p,q)$.

(p,q)-envelope $\mathcal{E}^{p,q}$. First we introduce the (p,q)-envelope $\mathcal{E}^{p,q}$ for each pair (p,q). Let (p,q) be a pair of elements in $\mathcal{L}$. Let $I(p,q)$ be the set of elements u represented by the join of some (p',q') with $p \succeq p' \succeq p \wedge q \preceq q' \preceq q$. This representation $u = p' \vee q'$ is uniquely determined, where (p',q') is given as $(p',q') = (p \wedge u, q \wedge u)$ [1, 12]. Define the map $\varphi^{p,q} : I(p,q) \to \mathbf{R}^2_+$ by

$$\varphi^{p,q}(u) := (r[p \wedge q, p \wedge u], r[p \wedge q, q \wedge u]) \quad (u \in I(p,q)). \tag{2.1}$$

Let $\mathrm{Conv}\, I(p,q)$ denote the convex hull of $\varphi^{p,q}(I(p,q))$ in $\mathbf{R}^2_+$. The (p,q)-*envelope* $\mathcal{E}^{p,q}$ is the set of elements $u \in I(p,q)$ such that $\varphi^{p,q}(u)$ is a maximal extreme point of $\mathrm{Conv}\, I(p,q)$, where a maximal extreme point is an extreme point z in $\mathrm{Conv}\, I(p,q)$ such that every positive vector ϵ it holds $z + \epsilon \notin \mathrm{Conv}\, I(p,q)$. In fact, the map $\varphi^{p,q}$ is injective on $\mathcal{E}^{p,q}$ [12], and hence is a bijection between $\mathcal{E}^{p,q}$ and the set of maximal extreme points of $\mathrm{Conv}\, I(p,q)$.

Valuation ν of convex cones in $\mathbf{R}^2_+$. To define the coefficients ν_u in the fractional join, we need notions of cone subdivisions of $\mathbf{R}^2_+$ and a certain valuation of convex cones in $\mathbf{R}^2_+$. A *cone subdivision* of $\mathbf{R}^2_+$ is a set $\mathcal{C}$ of 2-dimensional (closed convex) cones such that distinct $C, C' \in \mathcal{C}$ have no common interior point, and the union of $\mathcal{C}$ is equal to $\mathbf{R}^2_+$. For cone subdivisions $\mathcal{C}, \mathcal{C}'$, the *common refinement* of $\mathcal{C}$ and $\mathcal{C}'$ is the set of 2-dimensional cones that is the intersection of a cone in $\mathcal{C}$ and a cone in $\mathcal{C}'$. The common refinement is also a cone subdivision of $\mathbf{R}^2_+$.

Every closed convex cone C in $\mathbf{R}^2_+$ is uniquely represented as

$$C = \{(x,y) \in \mathbf{R}^2_+ \mid -x \sin\alpha + y \cos\alpha \leq 0,\ -x \sin\beta + y \cos\beta \geq 0\}$$

for some $0 \leq \beta \leq \alpha \leq \pi/2$. Define $\nu(C)$ by

$$\nu(C) := \frac{\sin\alpha}{\cos\alpha + \sin\alpha} - \frac{\sin\beta}{\cos\beta + \sin\beta}. \tag{2.2}$$

Then one can see the following properties of ν:

(2.3) (1) $\nu(C) \geq 0$, and $\nu(C) > 0$ if C is full dimensional.

(2) $\nu(C) + \nu(C') = \nu(C \cap C') + \nu(C \cup C')$ for $C \cap C' \neq \emptyset$.

(3) $\nu(\mathbf{R}^2_+) = 1$.

Fractional joins. Let (p, q) be a pair of elements. Consider the (p, q)-envelope $\mathcal{E}^{p,q}$. For $u \in \mathcal{E}^{p,q}$, let $[u](= [u]^{p,q})$ denote the set of nonnegative vectors $w \in \mathbf{R}^2_+$ with $\langle w, \varphi(u)\rangle = \max_{u' \in \mathcal{E}^{p,q}} \langle w, \varphi(u')\rangle$, where $\langle , \rangle$ is the standard inner product. Then $[u]$ forms a closed convex cone in $\mathbf{R}^2_+$; $[u]$ is nothing but the intersection of $\mathbf{R}^2_+$ and the normal cone at $\varphi(u)$ of $\mathrm{Conv}\, I(p, q)$. The *fractional join* of (p, q) is a formal convex combination of $u \in \mathcal{E}^{p,q}$ with coefficient $\nu([u])$, i.e.,

$$\sum_{u \in \mathcal{E}^{p,q}} \nu([u])u.$$

We can also define the *fractional join operation* as follows. Let $\mathcal{N}^{p,q}_{\mathcal{L}} := \{[u] \mid u \in \mathcal{E}^{p,q}, [u] \text{ is 2-dimensional}\}$, which is a cone subdivision of $\mathbf{R}^2$. Let $\mathcal{N}_{\mathcal{L}}$ be the common refinement of $\mathcal{N}^{p,q}_{\mathcal{L}}$ over all $(p, q) \in \mathcal{L} \times \mathcal{L}$. For a cone C in $\mathcal{N}_{\mathcal{L}}$, we can define binary operation $\vee_C$ as follows: $p \vee_C q :=$ a unique element u in $\mathcal{E}^{p,q}$ with $C \subseteq [u]$. The *fractional join operation* is a formal sum of $\vee_C$ over $C \in \mathcal{N}$ with coefficient $\nu(C)$. Then, by definition and the valuation property (2.3), we obtain

$$\sum_{u \in \mathcal{E}^{p,q}} \nu([u])u = \sum_{C \in \mathcal{N}_{\mathcal{L}}} \nu(C)(p \vee_C q) \quad (p, q \in \mathcal{L}). \tag{2.4}$$

2.2 Submodular functions

Definition 2.2. A function $f : \mathcal{L} \to \bar{\mathbf{R}}$ is called *submodular* if it satisfies

$$f(p) + f(q) \geq f(p \wedge q) + \sum_{u \in \mathcal{E}^{p,q}} \nu([u]^{p,q})f(u) \quad (p, q \in \mathcal{L}). \tag{2.5}$$

By (2.4), we can define a submodular function by a function f satisfying

$$f(p) + f(q) \geq f(p \wedge q) + \sum_{C \in \mathcal{N}_{\mathcal{L}}} \nu(C)f(p \vee_C q) \quad (p, q \in \mathcal{L}). \tag{2.6}$$

In the case where f takes a finite value, a small set of inequalities defines the submodularity. A pair (p, q) is called *antipodal* if $\mathcal{E}^{p,q} \subseteq \{p, q\}$, or equivalently, if $r[p', p]r[q', q] \geq r[p \wedge q, p']r[p \wedge q, q']$ for every (p', q') with $p \succeq p' \succeq p \wedge q \preceq q' \preceq q$.

Theorem 2.3 ([12]). *$f : \mathcal{L} \to \mathbf{R}$ is submodular on $\mathcal{L}$ if and only if it satisfies*

$$f(p) + f(q) \geq f(p \wedge q) + f(p \vee q)$$

for every bounded pair (p, q) and

$$r[p \wedge q, q]f(p) + r[p \wedge q, p]f(q) \geq (r[p \wedge q, p] + r[p \wedge q, q])f(p \wedge q)$$

for every antipodal pair (p, q).

In many cases, a modular semilattice in question is the n product $\mathcal{L}^n$ of a small modular semilattice $\mathcal{L}$; the direct product of modular semilattice is also a modular semilattice. For $C \in \mathcal{N}_{\mathcal{L}}$, extend operation $\vee_C : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ to $\mathcal{L}^n \times \mathcal{L}^n \to \mathcal{L}^n$ by $(p_1, p_2, \ldots, p_n) \vee_C (q_1, q_2, \ldots, q_n) := (p_1 \vee_C q_1, p_2 \vee_C q_2, \ldots, p_n \vee_C q_n)$.

Theorem 2.4 ([12])**.** *f is submodular on $\mathcal{L}^n$ if and only if f satisfies* (2.6).

Submodularity with respect to a valuation. A *valuation* on $\mathcal{L}$ is a function $v : \mathcal{L} \to \mathbf{R}$ satisfying $v(p) < v(q)$ for $p \prec q$ and $v(p) + v(q) = v(p \wedge q) + v(p \vee q)$ for any bounded pair p, q. The rank function r is a valuation. In the definitions of $\varphi^{p,q}$, we can use any valuation v instead of r. Hence we call a function satisfying the condition in Definition 2.2 with r being replaced by v a *submodular function with respect to v*. We will see a particular example in Section 2.4.

2.3 Tractability under VCSP model

Consider the minimization problem on a submodular function f on the n product $\mathcal{L}^n$ on modular semilattice $\mathcal{L}$. This problem is tractable if the problem is represented as a valued CSP form, though we do not know whether f can be minimized by calling an oracle of f in time polynomial of n and $|\mathcal{L}|$.

Let D be a finite set. A *cost function* f on D is a function from D^k to $\bar{\mathbf{Q}}$ for some positive integer $k = \mathrm{ar}(f)$, called the *arity* of f. A (valued-constraint) *language* is a finite set Ω of cost functions on D. *Valued CSP (Valued Constraint Satisfaction Problem)* on Ω is:

VCSP$[\Omega]$**:** Given $n \in \mathbf{Z}_+$ and $c^f_{i_1,i_2,\ldots,i_{\mathrm{ar}(f)}} \in \mathbf{Q}_+$ $(f \in \Omega, 1 \leq i_1 < i_2 < \cdots < i_{\mathrm{ar}(f)} \leq n)$

$$\text{Minimize} \quad \sum_{f \in \Omega;\ i_1,i_2,\ldots,i_{\mathrm{ar}(f)}} c^f_{i_1,i_2,\ldots,i_{\mathrm{ar}(f)}} f(x_{i_1}, x_{i_2}, \ldots, x_{i_{\mathrm{ar}(f)}}) \quad \text{over} \quad (x_1, x_2, \ldots, x_n) \in D^n.$$

See [32] for a formal treatment of valued CSP. In this model, the size of input is $O(|\Omega| n^K B)$, where K is the maximum arity of a cost function in Ω, and B is a bit length representing c and f. By a polynomial of the input we mean a polynomial of $|\Omega| n^K$ (and B). Then **VCSP**$[\Omega]$ has an IP formulation of polynomial size. This gives the following natural LP relaxation, called the *Basic-LP*.

$$\begin{array}{lll}
\text{Basic-LP:} & \text{Min.} & \displaystyle\sum_{f \in \Omega;\ i_1,i_2,\ldots,i_{\mathrm{ar}(f)}} \sum_{p_i \in D} c^f_{i_1,i_2,\ldots,i_{\mathrm{ar}(f)}} f(p_1, p_2, \ldots, p_{\mathrm{ar}(f)}) \lambda^{f;p_1,p_2,\ldots,p_{\mathrm{ar}(f)}}_{i_1,i_2,\ldots,i_{\mathrm{ar}(f)}} \\
& \text{s.t.} & \displaystyle\sum_{\exists k, (i_k, p_k) = (i, p)} \lambda^{f;p_1,p_2,\ldots,p_{\mathrm{ar}(f)}}_{i_1,i_2,\ldots,i_{\mathrm{ar}(f)}} = \lambda^p_i \quad (f \in \Lambda, 1 \leq i \leq n, p \in D), \\
& & \displaystyle\sum_{p \in D} \lambda^p_i = 1 \quad (1 \leq i \leq n), \\
& & \lambda^{f;p_1,p_2,\ldots,p_{\mathrm{ar}(f)}}_{i_1,i_2,\ldots,i_{\mathrm{ar}(f)}} \geq 0,\ \lambda^p_i \geq 0.
\end{array}$$

Recently, Thapper and Živný [30] discovered a surprising criterion for which the Basic-LP is exact, and consequently **VCSP**$[\Omega]$ is solvable in polynomial time. Their criterion is described by the notion of fractional polymorphisms. An *operation* on D is a function from $D \times D$ to D. The set of all operations on D is denoted by $\mathcal{O}$. By component-wise action, we extend an operation on D to an operation on D^n. A *fractional polymorphism* for language Ω is a function $\omega : \mathcal{O} \to \mathbf{R}_+$ satisfying

$$\begin{aligned}
\sum_{g \in \mathcal{O}} \omega(g) &= 1, \\
\sum_{g \in \mathcal{O}} \omega(g) f(g(p,q)) &\leq \frac{1}{2} f(p) + \frac{1}{2} f(q) \quad (f \in \Omega, (p,q) \in D^{\mathrm{ar}(f)} \times D^{\mathrm{ar}(f)}).
\end{aligned}$$

Theorem 2.5 ([30]). *If Ω has a fractional polymorphism ω such that the nonzero support of ω includes a semilattice operation, then Basic-LP is exact, and* $\mathbf{VCSP}[\Omega]$ *is solvable in polynomial time.*

Here a semilattice operation is an operation g satisfying $g(p,q) = g(q,p)$, $g(p,p) = p$, $g(p, g(q,r)) = g(g(p,q), r)$ for $p,q,r \in D$. See [23, 31] for subsequent developments.

Consider the case where D is a modular semilattice $\mathcal{L}$. A language on $\mathcal{L}$ is *submodular* if each cost function is submodular (on the product of $\mathcal{L}$). Define $\omega_{\mathcal{L}} : \mathcal{O} \to \mathbf{R}_+$ by

$$\omega_{\mathcal{L}}(g) := \begin{cases} 1/2 & \text{if } g = \wedge, \\ \nu(C)/2 & \text{if } \exists C \in \mathcal{N}_{\mathcal{L}},\ g = \vee_C, \\ 0 & \text{otherwise,} \end{cases} \qquad (g \in \mathcal{O}). \tag{2.7}$$

Theorem 2.4 and the property (2.3) of ν imply:

Theorem 2.6 ([12]). *$\omega_{\mathcal{L}}$ is a fractional polymorphism for any submodular language on modular semilattice $\mathcal{L}$.*

Since $\wedge$ is a semilattice operation, from Theorem 2.5 we obtain:

Theorem 2.7 ([12]). *Suppose that Ω is a submodular language on modular semilattice $\mathcal{L}$. Then* $\mathbf{VCSP}[\Omega]$ *is solvable in polynomial time.*

In the case where $\mathcal{L}$ is the product of distinct modular semilattices $\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_n$, a sum of submodular functions of bounded arity is minimized by Basic-LP [12].

2.4 Examples

Here we explain relations between our submodular functions and other submodular-type functions. Consider the case where a modular semilattice $\mathcal{L}$ is a lattice. Then $\mathcal{L}$ is a modular lattice. For every pair (p,q), the fractional join of (p,q) is equal to $p \vee q$. Therefore our definition of submodular functions coincides with the ordinary definition of submodular functions on (modular) lattices.

Bisubmodular functions. Consider the 3-element poset $\mathcal{S}_2 := \{+, 0, -\}$ ordered by $+ \succ 0 \prec -$ with no relation between $+$ and $-$. Define binary operation $\sqcup$ on $\mathcal{S}_2$ by

$$x \sqcup y := \begin{cases} x \vee y & \text{if } x \preceq y \text{ or } y \preceq x, \\ 0 & \text{if } \{x,y\} = \{+,-\}. \end{cases} \tag{2.8}$$

Next consider the n product ${\mathcal{S}_2}^n$ of $\mathcal{S}_2$. Define $\sqcup$ by $(p_1, p_2, \ldots, p_n) \sqcup (q_1, q_2, \ldots, q_n) := (p_1 \sqcup q_1, p_2 \sqcup q_2, \ldots, p_n \sqcup q_n)$.

A function f on ${\mathcal{S}_2}^n$ is called *bisubmodular* if

$$f(p) + f(q) \geq f(p \wedge q) + f(p \sqcup q) \quad (p, q \in {\mathcal{S}_2}^n). \tag{2.9}$$

It is obvious that $\mathcal{S}_2$ is a modular semilattice, and so is its direct product.

Proposition 2.8 ([14]). *$f : {\mathcal{S}_2}^n \to \bar{\mathbf{R}}$ is bisubmodular if and only if f is submodular on modular semilattice ${\mathcal{S}_2}^n$.*

It should be noted that the set (2.5) of inequalities corresponding to ${\mathcal{S}_2}^n$ is *different* from (2.9). However they defines the same class of functions. The explicit description of the (p,q)-envelope and fractional joins is given by [14].

k-submodular functions. Huber and Kolmogorov [16] introduced the notion of k-submodular functions as a generalization of the bisubmodularity. Consider a $k+1$ element poset $\mathcal{S}_k$ having a special element 0, ordered by $0 \preceq x$ $x \in \mathcal{S}_k \setminus \{0\}$ with no relation among $\mathcal{S}_k \setminus \{0\}$. As above, define binary operation

$$x \sqcup y := \begin{cases} x \vee y & \text{if } x \preceq y \text{ or } y \preceq x, \\ 0 & \text{if } 0 \neq x \neq y \neq 0. \end{cases} \tag{2.10}$$

Consider the product ${\mathcal{S}_k}^n$, and extend $\sqcup$ to an operation on ${\mathcal{S}_k}^n$, as above. A function $f : {\mathcal{S}_k}^n \to \bar{\mathbf{R}}$ is called *k-submodular* if

$$f(p) + f(q) \geq f(p \wedge q) + f(p \sqcup q) \quad (p, q \in {\mathcal{S}_k}^n). \tag{2.11}$$

Again ${\mathcal{S}_k}^n$ is a modular semilattice. Then we have:

Proposition 2.9 ([14]). *$f : {\mathcal{S}_k}^n \to \bar{\mathbf{R}}$ is k-submodular if and only if f is submodular on modular semilattice ${S_k}^n$.*

α-bisubmodular functions. Huber, Krokhin, and Powell [15] introduced the notion of α-bisubmodular function on ${\mathcal{S}_2}^n$ for classifying tractable VCSP on 3-element set. In addition to $\sqcup$, define a new binary operation $\sqcup_+$ on $\mathcal{S}_2 = \{-, 0, +\}$ by

$$x \sqcup_+ y = \begin{cases} + & \text{if } \{x, y\} = \{-, +\}, \\ x \sqcup y & \text{otherwise}, \end{cases} \tag{2.12}$$

and extend it to an operation on ${\mathcal{S}_2}^n$. For $\alpha \in (0, 1]$, a function $f : {\mathcal{S}_2}^n \to \mathbf{R}$ is called *α-bisubmodular* (toward +) if

$$f(p) + f(q) \geq f(p \wedge q) + \alpha f(p \sqcup q) + (1 - \alpha) f(p \sqcup_+ q) \quad (p, q \in {\mathcal{S}_2}^n). \tag{2.13}$$

In particular, 1-bisubmodularity is bisubmodularity.

Our framework also includes α-bisubmodularity. Recall the end of Section 2.2 for the submodularity with respect to a valuation. For $\alpha \in (0, 1]$, define a valuation of $\mathcal{S}_2$ by

$$v_\alpha(0) := 0,\ v_\alpha(+) := 1,\ v_\alpha(-) := 2\alpha/(1 + \alpha). \tag{2.14}$$

Then v_α is extended to a valuation of ${\mathcal{S}_2}^n$.

Proposition 2.10 ([14]). *$f : {\mathcal{S}_2}^n \to \bar{\mathbf{R}}$ is α-bisubmodular if and only if f is submodular on modular semilattice ${\mathcal{S}_2}^n$ with respect to valuation v_α.*

3 L-convex functions on modular complexes

In this section, we explain a theory of L-convex functions on modular complexes. In Section 3.1, we explain several structural properties of a modular complex. In Section 3.2, we introduce L-convex functions on a modular complex, and present their intriguing minimization properties. In Section 3.3, we discuss examples. L$^\natural$-convex functions by Murota [27], strongly tree-submodular functions by Kolmogorov [22], and UJ-convex functions by Fujishige [6] can be understood as L-convex functions on special modular complexes.

3.1 Modular complexes

Let Γ be an orientable modular graph with an admissible orientation o. A pair (Γ, o) is called a *modular complex*. See Figures 2 for a modular complex. In this section we show that a modular complex has a subdivision operation, which enable us to consider a kind of the neighborhood around each vertex; see Figure 3.

Since an admissible orientation is acyclic, o induces a partial order $\preceq$ on V_Γ. Then every interval is a modular lattice.

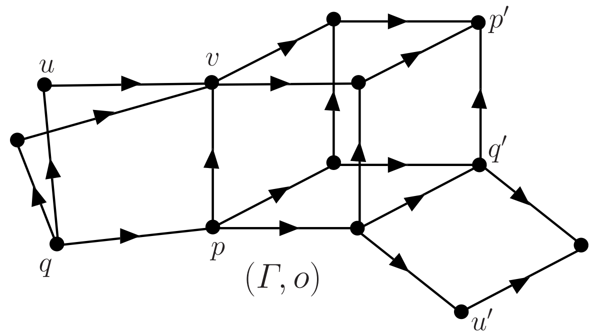

Figure 2: Modular complex

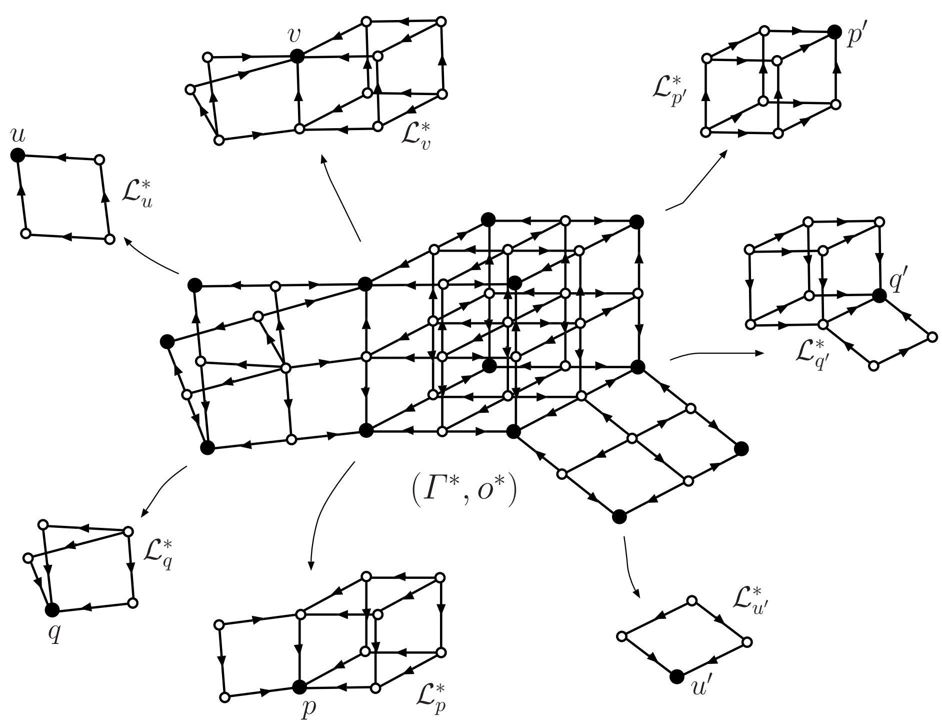

Figure 3: 2-subdivision and neighborhood semilattices

Lemma 3.1 ([12]). *If $p \preceq q$, then $[p,q]$ is a modular lattice.*

We are interested in special intervals. A pair (p,q) is called *Boolean* if $p \preceq q$ and $[p,q]$ is a geometric modular lattice, where a modular semilattice is *geometric* (or *complemented*) if every element is a join of atoms (rank 1 elements). We define relation $\sqsubseteq$ by

$$p \sqsubseteq q \Leftrightarrow (p,q) \text{ is a Boolean pair.}$$

For a Boolean pair (p,q), any (p',q') with $p \preceq p' \preceq q' \preceq q$ is Boolean. Note that the relation $\sqsubseteq$ is not transitive. A Boolean pair (p,q) is also denote by q/p. For a vertex p, define subset $\mathcal{L}_p^+$ by

$$\mathcal{L}_p^+ := \{q \in V_\Gamma \mid p \sqsubseteq q\}, \quad \mathcal{L}_p^- := \{q \in V_\Gamma \mid q \sqsubseteq p\}. \tag{3.1}$$

We regard $\mathcal{L}_p^+$ as a poset with the partial order induced by $\sqsubseteq$, and $\mathcal{L}_p^-$ as a poset with the partial order induced by the reverse of $\sqsubseteq$.

Proposition 3.2 ([12]). *For every vertex p, $\mathcal{L}_p^+$ and $\mathcal{L}_p^-$ are geometric modular semilattices with the lowest element p.*

The 2-*subdivision* Γ^* of Γ is a simple undirected graph on the set of all Boolean pairs with edges given as: q/p and q'/p' are adjacent if and only if $p = p'$ and $qq' \in E_\Gamma$ or $q = q'$ and $pp' \in E_\Gamma$. The orientation o^* for Γ^* is given as: $q'/p' \to q/p$ if $p = p'$ and $q' \to q$ in o, or $q = q'$ and $p \to p'$ in o.

Theorem 3.3 ([12]). *The 2-subdivision Γ^* is orientable modular and orientation o^* is admissible.*

Hence (Γ^*, o^*) is also a modular complex, which is called the 2-*subdivision* of (Γ, o). By embedding $p \mapsto (p,p)$, we can regard $V_\Gamma \subseteq V_{\Gamma^*}$. The admissible orientation o^* is oriented so that the vertices in V_Γ are all sinks. For each vertex $p \in V_\Gamma$, define the *neighborhood semilattice* $\mathcal{L}_p^* := \mathcal{L}_p^+(\Gamma^*, o^*)$, which is a geometric modular semilattice with the lowest element p

We introduce a special class of modular complexes, which will turn out to be fundamental and useful. A modular complex (Γ, o) is called *transitive* if $\sqsubseteq$ is transitive, or equivalently, if every pair $p, q \in V_\Gamma$ with $p \preceq q$ is Boolean.

Lemma 3.4 ([13]). *The 2-subdivision (Γ^*, o^*) is transitive.*

3.2 L-convex functions

Next we introduce the concept of L-convex functions for a modular complex (Γ, o). Consider the 2-subdivision (Γ^*, o^*) of (Γ, o). For a function $g : V_\Gamma \to \bar{\mathbf{R}}$, define $\bar{g} : V_{\Gamma^*} \to \bar{\mathbf{R}}$ by

$$\bar{g}(q/p) := \frac{g(p) + g(q)}{2} \quad (q/p \in V_{\Gamma^*}). \tag{3.2}$$

A set B of vertices is $\sqsubseteq$-*connected* if for every $p, q \in B$ there is a sequence $(p = p_0, p_1, p_2, \ldots, p_n = q)$ with $p_i \sqsubseteq p_{i+1}$ or $p_{i+1} \sqsubseteq p_i$ for $i = 0, 1, 2, \ldots, n-1$.

Definition 3.5. A function $g : V_\Gamma \to \bar{\mathbf{R}}$ is *L-convex* on (Γ, o) if $\operatorname{dom} g$ is $\sqsubseteq$-connected, and $\bar{g}$ is submodular on $\mathcal{L}_p^*$ for every $p \in \operatorname{dom} g$.

An L-convex function is a submodular function on the interval formed each Boolean pair.

Lemma 3.6 ([12]). *If g is L-convex, then g is submodular on $\mathcal{L}_p^+$ and on $\mathcal{L}_p^-$ for every vertex $p \in \operatorname{dom} g$.*

In the case of transitive modular complexes, the converse holds.

Proposition 3.7 ([14]). *Suppose that (Γ, o) is transitive. Then g is L-convex on (Γ, o) if and only if g is submodular on $\mathcal{L}_p^+$ and on $\mathcal{L}_p^-$ for every $p \in \operatorname{dom} g$.*

Once a minimizer of $\bar{g}$ over (Γ^*, o^*) is obtained, so is a minimizer of g over (Γ, o). The following property says that, in a sense, it suffices to consider L-convex functions on transitive modular complexes.

Lemma 3.8 ([13]). *If g is an L-convex function on (Γ, o), then $\bar{g}$ is an L-convex function on (Γ^*, o^*).*

L-optimality criterion and steepest descent algorithm. We consider the minimization of L-convex functions. As in the case of L$^\natural$-convex functions, a local optimality guarantees the global optimality, and the local optimality is checked by submodular function minimizations.

Theorem 3.9 ([12])**.** *For an L-convex function g on (Γ, o) and $p \in V_\Gamma$, the following conditions are equivalent:*

(1) $g(p) = \min\{g(q) \mid q \in V_\Gamma\}$.

(2) $g(p) = \min\{g(q) \mid q \in \mathcal{L}_p^+ \cup \mathcal{L}_p^-\}$.

The condition (2) can be checked by minimizing g over $\mathcal{L}_p^+$ and g over $\mathcal{L}_p^-$. This leads us to the following descent algorithm, which is a straightforward generalization of the steepest descent algorithm of L$^\natural$-convex functions [27].

Steepest Descent Algorithm

Input: An L-convex function g and a vertex $p \in V_\Gamma$.

step 1: Let q be a minimizer of g over $\mathcal{L}_p^- \cup \mathcal{L}_p^+$

step 2: If $g(p) = g(q)$, then p is optimal, and stop.

step 3: $p := q$, and go to step 1.

In applications, a modular complex in question is the product of small modular complexes (Γ_i, o_i) $(i = 1, 2, \ldots, n)$, and L-convex function g is a sum of L-convex functions g_j of small arity. Then step 1 is conducted by minimizing a sum of submodular functions of bounded arity, which can be efficiently solved by the argument in Section 2.3.

We need to estimate the number of iteration of the algorithm to obtain a complexity bound. In the case of L$^\natural$-convex functions, the number of iterations is bounded by the l_∞-diameter of the effective domain. We do not know whether a similar bound holds for our cases in general. However, in the case where (Γ, o) is transitive, an analogous bound holds.

Let Γ^Δ be the graph on the vertex set V_Γ with the edges given as

$$pq \in E_{\Gamma^\Delta} \Leftrightarrow (p \wedge q, p \vee q) \text{ is a Boolean pair.} \tag{3.3}$$

Since $p \sqsubseteq q$ implies $pq \in E_{\Gamma^\Delta}$, the steepest descent algorithm yields a path in Γ^Δ. Let $\mathrm{opt}(g)$ be the set of minimizers of g. Obviously the total number of the iterations (= the length of the path in Γ^Δ) is at least $d_{\Gamma^\Delta}(p, \mathrm{opt}(g)) = \min_{q \in \mathrm{opt}(g)} d_{\Gamma^\Delta}(p, q)$. This lower bound is attained when (Γ, o) is transitive, and the initial point p is a sink or a source, where a sink is a vertex p with $\mathcal{L}_p^- = \{p\}$ and a source is a vertex p with $\mathcal{L}_p^+ = \{p\}$.

Theorem 3.10 ([13])**.** *Suppose that (Γ, o) is transitive, and p is a sink or a source. The total number of the iterations of the steepest descent algorithm is equal to $d_{\Gamma^\Delta}(p, \mathrm{opt}(g))$.*

In particular, the path produced by the algorithm is a geodesic from p to $\mathrm{opt}(g)$; the proof needs a deep geometric investigation of the graph Γ^Δ [13].

The assumption of Theorem 3.10 is not restrictive. Recall Lemmas 3.4 and 3.8. To guarantee the iteration bound in Theorem 3.10, simply apply the steepest descent algorithm to $\bar{g}$ on (Γ^*, o^*) instead of g on (Γ, o). Furthermore we can conduct this without an explicit reference of the 2-subdivision. This leads us to the following variation of the steepest descent algorithm, called the *geodesic descent algorithm.*

Geodesic Descent Algorithm

Input: An L-convex function g and a vertex $p \in V_\Gamma$.

step 1: Let q^- and q^+ be minimizers of g over $\mathcal{L}_p^-$ and $\mathcal{L}_p^+$, respectively, with the property $q^- \sqsubseteq q^+$.

step 2: Let q be a minimizer of g over $[q^-, q^+]$.

step 3: If $g(p) = g(q)$, then p is optimal, and stop.

step 4: $p := q$, and go to step 1.

By a simple application of inequality (2.5), one can see that a pair (q^-, q^+) of minimizers in step 1 always exists. This algorithm emulates the steepest descent algorithm for $\bar{g}$; q^+/q^- is a steepest direction at p/p, and q/q is a next steepest direction at q^+/q^-. Hence we get the following.

Corollary 3.11 ([13]). *The total number of the iterations of the geodesic descent algorithm is equal to* $d_{\Gamma^{\triangle}}(p, \mathrm{opt}(g))$.

3.3 Examples

Let T be a tree with an orientation. We call the resulting modular complex an *oriented tree.* Then we can define binary operations m^- and m^+ (*midpoint operators*) as follows. For $p, q \in V_T$ there uniquely exists a pair (u, v) of vertices such that $d_T(p, u) = d_T(u, q) = \lfloor d_T(p, q)/2 \rfloor$ and $d_T(p, q) = d_T(p, u) + d_T(u, v) + d_T(v, q)$. Then $d_T(u, v) = 0$ or 1. Define m^- and m^+ by

$$(m^-(p, q), m^+(p, q)) := \begin{cases} (u, v) & \text{if } u \preceq v \\ (v, u) & \text{if } v \preceq u \end{cases} \tag{3.4}$$

This definition is a straightforward generation of one given by Kolmogorov [22] for the case where the tree has a root.

Consider the product (Γ, o) of oriented trees. By component-wise actions of m^+ and m^-, we obtain operations on V_Γ which are also denoted by m^+ and m^-, respectively.

Proposition 3.12 ([14]). $g : V_\Gamma \to \bar{\mathbf{R}}$ *is L-convex on the product* (Γ, o) *of oriented trees if and only if it satisfies*

$$g(p) + g(q) \geq g(m^+(p, q)) + g(m^-(p, q)) \quad (p, q \in V_\Gamma). \tag{3.5}$$

In the case where each oriented tree has a root vertex, our submodular function coincides with Kolmogorov's strongly tree-submodular function [23], which is defined to be a function satisfying (3.5). Hence we obtain the following.

Proposition 3.13 ([14]). *g is strongly tree-submodular if and only if g is L-convex on the product of rooted trees.*

L$^\natural$-convex functions. A *linear n-grid* is a pair (Gr^n, o) of an n-grid Gr^n and a linear orientation o. Then (Gr^n, o) is a modular complex, and is the n product of directed paths. Recall that a linear n-grid is identified with $\mathbf{Z}^n$. Then (3.5) coincides with the discrete midpoint convexity (1.1).

Proposition 3.14 ([14]). $g : \mathbf{Z}^n \to \bar{\mathbf{R}}$ *is L$^\natural$-convex if and only if g is L-convex on the linear n-grid.*

UJ-convex functions. Fujishige [6] introduced a *UJ-convex function* as a function g on $\mathbf{Z}^n$ such that the piecewise-linear interpolation $\bar{g}$ of g with respect to the Union-Jack subdivision is convex on $\mathbf{R}^n$. The *Union-Jack subdivision* is a triangulation of $\mathbf{Z}^n$ each of whose full dimensional simplexes is the convex hull of

$$x + \sum_{k=1}^{j} \sigma_{i_k} e_{i_k} \quad (j = 0, 1, 2, \ldots, n)$$

for some *even* vector $x \in (2\mathbf{Z})^n$, $\{-1, 1\}$-vector $\sigma \in \{-1, 1\}^n$, and permutation $(i_1, i_2, \ldots, i_n)$ of $\{1, 2, \ldots, n\}$, where e_i is the i-th unit vector.

Identify $\mathbf{Z}^n$ with (the vertex set of) n-grid Gr_n. Consider the following orientation o': $x \to x + e_i$ if x_i is odd and $x + e_i \to x$ otherwise. Orientation o' is admissible. The resulting modular complex (Gr_n, o) is called an *alternating n-grid.*

Proposition 3.15 ([14]). $g : \mathbf{Z}^n \to \bar{\mathbf{R}}$ *is UJ-convex if and only if g is L-convex on the alternating n-grid.*

4 Multicommodity flows and 0-extensions

The motivations of our theory come from combinatorial dualties of multicommodity flow problems and the tractability classification of the minimum 0-extension problem. In both applications, L-convex functions arise as a weighted sum of metric functions of a modular complex. A *binary function* is a function of arity 2; a metric function is binary.

Minimum 0-extension problems. Let (S, μ) be a finite metric space. The *minimum 0-extension problem* on (S, μ) is: for given a set V including S and a nonnegative weight c on $\binom{V}{2}$, find a 0-extension (V, d) of (S, μ) with minimum cost $\sum_{xy} c(xy)d(x, y)$, where 0-*extension* (V, d) of (S, μ) is a (semi)metric space such that metric d is represented as the composition $\mu \circ \rho$ for some retraction $\rho : V \to S$.

Consider the case where (S, μ) is the path metric space (V_Γ, d_Γ) of a graph Γ. One can see that the minimum 0-extension problem is equivalent to the following *facility location* problem on Γ:

0-Ext$[\Gamma]$: Given integer $n > 0$ and nonnegative weights b_i^s, c_{ij} $(s \in V_\Gamma, 1 \leq i \leq n)$,

$$\text{Minimize} \quad \sum_{s \in V_\Gamma, 1 \leq i \leq n} b_i^s d_\Gamma(s, p_i) + \sum_{1 \leq i < j \leq n} c_{ij} d_\Gamma(p_i, p_j)$$

$$\text{over} \quad (p_1, p_2, \ldots, p_n) \in V_\Gamma \times V_\Gamma \times \cdots \times V_\Gamma.$$

Note that **0-Ext**$[\Gamma]$ is a valued CSP for language $\Omega := \{d_\Gamma\} \cup \{d_\Gamma(s, \cdot) \mid s \in V_\Gamma\}$. The minimum 0-extension problem includes the minimum cut problem for $\Gamma = K_2$, as well as multiway cut problem for $\Gamma = K_n$ $(n > 2)$. The former problem is tractable, and the latter is NP-hard. Karzanov raised a question: *What are Γ for which* **0-Ext**$[\Gamma]$ *is tractable ?* He proved the NP-hardness for every graph which is not modular or not orientable.

Theorem 4.1 ([19]). *If Γ is not modular or not orientable, then* **0-Ext**$[\Gamma]$ *is NP-hard.*

This result implies that a tractable graph is necessarily orientable and modular (unless P = NP). Previous researches have shown the tractability for special classes of orientable modular graphs: trees [28], median graphs [2], and orientable hereditary modular graphs [19].

We formulate **0-Ext**$[\Gamma]$ as an optimization over the product $\bar{\Gamma} := \Gamma \times \Gamma \times \cdots \times \Gamma$ of Γ. If Γ is an orientable modular graph with admissible orientation o, then the product $\bar{\Gamma}$ is also an orientable modular graph with admissible orientation $\bar{o} := o \times o \times \cdots \times o$.

Theorem 4.2 ([12]). *Let Γ be an orientable modular graph with admissible orientation o. The metric function d_Γ is L-convex on $(\Gamma \times \Gamma, o \times o)$, and hence* **0-Ext**$[\Gamma]$ *is a minimization of a sum of binary L-convex functions on $(\bar{\Gamma}, \bar{o})$.*

Consequently we obtain the converse of Theorem 4.1.

Theorem 4.3 ([12]). *If Γ is orientable modular, then* **0-Ext**$[\Gamma]$ *is solvable in polynomial time.*

This complexity dichotomy of the minimum 0-extension problem can be understood as a special case of a more general dichotomy theorem of finite-valued CSP [31].

Multicommodity flows. A large class of tractable 0-extension problems arises from weighted maximum multicommodity flow problem. Let $G = (V, E)$ be a complete undirected graph with terminal set $S \subseteq V$ and nonnegative edge-capacity c. Suppose that $V = \{1, 2, \ldots, n\}$ and $S = \{1, 2, \ldots, k\}$. For distinct vertices i, j, the edge-capacity on edge ij is denoted by c_{ij}. Let $\mathcal{P}$ be the set of all paths whose ends are distinct terminals in S. A *multicommodity flow* (*multiflow* for short) is a nonnegative-valued function f on $\mathcal{P}$ satisfying the capacity constraint:

$$\sum \{f(P) \mid P \in \mathcal{P}, ij \in P\} \leq c_{ij} \quad (1 \leq i < j \leq n)$$

We are given a weight function μ defined on the set of all pairs on terminal set S. For a multiflow f, the μ-value $\mu \circ f$ is defined by $\sum_{P \in \mathcal{P}} \mu(s_P, t_P) f(P)$, where s_p and t_P are the ends of P. The μ-weighted maximum multiflow problem μ-MFP is:

$$\text{Maximize} \quad \mu \circ f \quad \text{over all multiflows } f \text{ on } (V, E, c, S) \tag{4.1}$$

In [8], we developed a duality theory of μ-MFP by using the *tight span* T_μ of μ [4, 17], which is defined to be the set of points p in $\mathbf{R}_+^S$ satisfying

$$p(i) = \max_{j \in S} \{\mu(i,j) - p(j)\} \quad (i \in S).$$

For a terminal $i \in S$, T_μ^i is the set of points p in T_μ with $p(i) = 0$. Consider the following continuous location problem on T_μ:

$$\begin{array}{lll} T\text{-Dual:} & \text{Minimize} & \displaystyle\sum_{1 \leq i \leq k} \delta_{T_\mu^i}(p_i) + \sum_{1 \leq i < j \leq n} c_{ij} \|p_i - p_j\|_\infty \\ & \text{over} & (p_1, p_2, \ldots, p_n) \in T_\mu \times T_\mu \times \cdots \times T_\mu. \end{array}$$

Here δ_X is the indicator function of X, i.e., $\delta_X = 0$ if $p \in X$ and $\delta_X = \infty$ otherwise.

Theorem 4.4 ([8, 20])**.** *The maximum value of μ-MFP is equal to the minimum value of T-Dual.*

Let V_μ be the set of 1/4-integral points z in T_μ with $z(i) + z(j) \in 1/2\mathbf{Z}$. Consider the graph Γ_μ on V_μ by joining edge zw if $\|z - w\|_\infty = 1/4$ [8]. Define (uniform) edge-length by 1/4. Then $d_{\Gamma_\mu}(p,q) = \|p - q\|_\infty$ for $p, q \in V_\mu$. Define subset B_μ^i of V_μ by $T_\mu^i \cap V_\mu$. Restricting T_μ to V_μ, we obtain the following discrete location problem on Γ_μ.

$$\begin{array}{lll} \text{Discrete } T\text{-Dual:} & \text{Minimize} & \displaystyle\sum_{1 \leq i \leq k} \delta_{B_\mu^i}(p_i) + \sum_{1 \leq i < j \leq n} c_{ij} d_{\Gamma_\mu}(p_i, p_j) \\ & \text{over} & (p_1, p_2, \ldots, p_n) \in V_\mu \times V_\mu \times \cdots \times V_\mu. \end{array}$$

In the case where $\dim T_\mu \leq 2$, Discrete T-Dual attains an optimum of T-Dual.

Theorem 4.5 ([8, 20])**.** *Suppose that* $\dim T_\mu \leq 2$*, The maximum value of μ-MFP is equal to the minimum value of Discrete T-Dual.*

In fact, the graph Γ_μ is an orientable (hereditary) modular graph (of some admissible orientation o), and hence Discrete T-Dual is an L-convex function minimization on the product $\bar{\Gamma}_\mu$ of Γ_μ.

Theorem 4.6 ([14])**.** *Suppose that* $\dim T_\mu \leq 2$*. Discrete T-Dual is a minimization of a sum of binary L-convex functions on the modular complex $(\bar{\Gamma}_\mu, \bar{o})$.*

This theorem might be regarded as a multiflow analogue of: *the minimum cut problem, a dual of the maximum flow problem, is a submodular function minimization.* More of applications of our theory to the multiflow theory will be given in [14].

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