Multiflow Feasibility Problem for $K_3 + K_3$

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The 20th ISMP,
August 23 - 28, 2009, Chicago, USA
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Multiflows (Multicommodity flows)

\( G \): an undirected graph (supply graph)
\( c : EG \to \mathbb{R}_+ \): nonnegative edge capacity
\( S \subseteq VG \): terminal set

A multiflow \( f = (\mathcal{P}, \lambda) \) \( \iff \)
\( \mathcal{P} \): set of \( S \)-paths
\( \lambda : \mathcal{P} \to \mathbb{R}_+ \): flow-value function satisfying capacity constraint

\[
\sum\{\lambda(P) \mid P \in \mathcal{P} : e \in P\} \leq c(e) \quad (e \in EG).
\]
Multiflow feasibility problem

$H$: demand graph with $VH = S$

$q : EH \rightarrow \mathbb{R}_+:$ demand function on edges $EH$.

Find a multiflow $f = (\mathcal{P}, \lambda)$ satisfying demand requirement

$$\sum\{\lambda(P) \mid P \in \mathcal{P} : P \text{ is (s,t)-path}\} = q(st) \quad (st \in EH),$$

or establish that no such a multiflow exists.

We are interested in behavior of multiflows for a fixed $H$ and arbitrary $G, c, q$. 
• $H = K_2$: single commodity flows

**Theorem** [Ford-Fulkerson 54]

$c, q$ integral, feasible $\Rightarrow \exists$ integral solution.

• $H = K_2 + K_2$: 2-commodity flows

**Theorem** [Hu 63]

$c, q$ integral, feasible $\Rightarrow \exists$ half-integral solution.

• $H = K_2 + K_2 + \cdots + K_2$: $k$-commodity flows

??? $c, q$ integral, feasible $\Rightarrow \exists 1/p$-integral solution ($p \leq k$)???

(Jewell 67, Seymour 81)
Theorem [Lomonosov 85]
There is no integer \( k > 0 \) such that every feasible 3-commodity flow problem with integer capacity and demand has a \( 1/k \)-integral solution.

Fractionality
\[ \text{frac}(H) := \text{the least positive integer } k \text{ with the property:} \]
\[ \forall c, q \text{ integral, feasible } \Rightarrow \exists 1/k \text{-integral solution.} \]

Problem [Karzanov 89,90]
Classify demand graphs \( H \) with \( \text{frac}(H) < +\infty \).

Remark \( H \supseteq K_2 + K_2 + K_2 \Rightarrow \text{frac}(H) = +\infty \).
Demand graphs without $K_2 + K_2 + K_2$

(I) $H = K_4, C_5$, or star + star:

Theorem [Rothschild & Winston 66, Seymour 80, Lomonosov 76,85]

$c, q$ Eulerian, feasible $\Rightarrow \exists$ integral solution ($\rightarrow \frac{1}{\text{frac}(H)} = 2$).

by "$c, q$ Eulerian" we mean $(G + H, c + q)$ is Eulerian.

(II) $H = K_5$ or star + $K_3$:

Theorem [Karzanov 87]

$c, q$ Eulerian, feasible $\Rightarrow \exists$ integral solution ($\rightarrow \frac{1}{\text{frac}(H)} = 2$).

(III) $H = K_3 + K_3$ ....?
(III) $H = K_3 + K_3$

An example from A. Schrijver: *Combinatorial Optimization*, p. 1275.

$$H = \begin{array}{c}
\begin{array}{c}
\text{1} \\
\text{1}
\end{array}
\end{array}$$

$\rightarrow \frac{K_3 + K_3}{2} \geq 4$.

**Conjecture** [Karzanov 90, ICM, Kyoto]

1. $\frac{K_3 + K_3}{2} < +\infty$.

2. $c, q$ Eulerian, feasible $\Rightarrow \exists$ half-integral solution ($\rightarrow \frac{K_3 + K_3}{2} = 4$)

Cf. Problems 51, 52 in Schrijver’s book
Main Theorem [H. 08]

\[ H \equiv K_3 + K_3, \ c, q \text{ Eulerian, feasible} \]

\[ \Rightarrow \exists \ 1/12\text{-integral solution.} \]

\[ \rightarrow \text{the complete classification of} \]

\[ \text{demand graphs having finite fractionality} \]

Corollary \( \frac{1}{\text{frac}(H)} < +\infty \iff H \notin K_2 + K_2 + K_2 \).

Corollary \( \frac{1}{\text{frac}(K_3 + K_3)} \in \{4, 8, 12, 24\} \).
Proof Sketch (very rough)

1. reduction to $K_{3,3}$-metric weighted multiflow maximization

2. combinatorial duality relation (of max-flow min-cut type)

3. fractional splitting-off and potential update

Recall: splitting-off
**$K_{3,3}$-metric weighted maximum multiflow problem:**

$S \subseteq VG$: 6-element terminal set with $S = VK_{3,3}$.

Maximize $\sum_{P \in \mathcal{P}} d_{K_{3,3}}(s_P, t_P) \lambda(P)$ s.t. $f = (\mathcal{P}, \lambda)$ for $(G, c; S)$,

\[\Rightarrow \exists 1/12\text{-integral solution in Eulerian } K_3 + K_3\text{-feasibility problem}\]

**Remark** no integral optimum even if $(G, c)$ inner Eulerian.

**Theorem** [H. 08]

$\exists 1/12\text{-integral optimum in every inner Eulerian } K_{3,3}\text{-max problem.}$

$\Rightarrow \exists 1/12\text{-integral solution in Eulerian } K_3 + K_3\text{-feasibility problem}$
Combinatorial duality relation [Karzanov 89, 98]

\[
\begin{align*}
\text{Max.} & \quad \sum_{P \in \mathcal{P}} d_{K_{3,3}}(s_P, t_P) \lambda(P) \quad \text{s.t.} \quad f = (\mathcal{P}, \lambda) \text{ for } (G, c; S) \\
= \text{Min.} & \quad \sum_{xy \in EG} c(xy) d_{\Gamma_{3,3}, \frac{1}{2}}(\rho(x), \rho(y)) \\
\text{s. t.} & \quad \rho : VG \to V\Gamma_{3,3}, \quad \rho|_S = id
\end{align*}
\]

\(\leftarrow\) potential
Proposition \((G, c)\) inner Eulerian, \(y\) inner node in \(S_\rho\)
\Rightarrow \(y\) is splittable. \hspace{1cm} \text{(the proof based on [Karzanov 87])}

Corollary \(M_\rho \cup C_\rho = \emptyset \Rightarrow \exists \) integral optimum.

Main step: \((G, c)\): inner Eulerian, \(\rho\): optimal potential
\(G, c; \rho \rightarrow \cdots \rightarrow (G', c'; \rho')\) until \(M_\rho' \cup C_\rho' = \emptyset\) keeping \((G', kc')\) inner Eulerian
Fractional splitting-off

\( \tau = (e, y, e') \): fork

Splitting capacity:

\[ \alpha(\tau) := \max \{ 0 \leq \alpha \leq 2 \mid \text{opt}(G, c) = \text{opt}(G^{\tau}, c - \alpha \chi_{e^{\tau}}) \} \]

\[ = \min \left\{ \frac{\langle c, d_{\Gamma_3,3} \circ \rho' \rangle - \text{opt}(G, c)}{d_{\Gamma_3,3}(\rho'(y), \rho'(y^{\tau}))} \mid \rho' : \text{potential with } \rho'(y) \neq \rho'(y^{\tau}) \right\}, \]

where \( \langle c, d_{\Gamma_3,3} \circ \rho' \rangle := \sum c(xy)d_{\Gamma_3,3}(\rho'(x), \rho'(y)) \).

Remark \( \tau \) is splittable if and only if \( \alpha(\tau) = 2 \).

\( \rho: \) optimal potential \( \Rightarrow \) we can take \( \rho' \) combinatorially close to \( \rho \).
Splitting-off with Potential UPdate (SPUP)

Lemma $\rho$: optimal potential, $(G, c)$: Eulerian at $M_\rho \cup C_\rho$, $\forall \tau$: fork at $C_\rho$, $\exists \rho'$ s.t.

$\rho'(x) \neq \rho(x)$ implies $\rho(x) \rightarrow \rho'(x)$ or $\rho(x) \rightarrow \rho'(x)$ and

$$\alpha(\tau) = \frac{\langle c, d_{\Gamma_{3,3}} \circ \rho' \rangle - \langle c, d_{\Gamma_{3,3}} \circ \rho \rangle}{d_{\Gamma_{3,3}}(\rho'(y), \rho'(y^\tau))} \in \left\{0, \frac{1}{2}, \frac{2}{3}, 1, \frac{4}{3}, \frac{6}{4}, 2\right\}.$$ 

In addition, if $C_\rho = \emptyset$ and $c$ is integral, then so does for $\exists \tau$ at $M_\rho$.

SPUP: $(G, c; \rho) \leftarrow (G^\tau, c - \alpha(\tau)\chi_{e^\tau}; \rho')$
Proposition
We can repeat SPUPs until $M_{\rho} \cup C_{\rho} = \emptyset$ keeping $(G, 12c)$ inner Eulerian.

Concluding remarks

- We do not know whether $1/12$ is tight.
- Recently we proved a generalized conjecture for $k = 12$:

  For a terminal weight $\mu : \left( \frac{S}{2} \right) \to \mathbb{R}_+$,

  \[
  \dim T_{\mu} \leq 2 \text{ if and only if there exists } k > 0 \text{ such that every Eulerian } \mu\text{-max problem has a } 1/k\text{-integral optimum},
  \]

  where $T_{\mu} := \text{Minimal } \{ p \in \mathbb{R}^S_+ | p(s) + p(t) \geq \mu(s, t) \ (s, t \in S) \}$

- Half-integral $\Gamma_{3,3}$-metric packing [H. 07, Combinatorica, to appear]