Multiflow Feasibility Problem for $K_3 + K_3$

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Contents


- Reviews on multiflow feasibility problems
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Multiflows (Multicommodity flows)

\( G \): an undirected graph \((\text{supply graph})\)

\( c : EG \to \mathbb{R}_+ \): nonnegative edge capacity

\( S \subseteq VG \): terminal set

A **multiflow** \( f = (\mathcal{P}, \lambda) \) \( \overset{\text{def}}{\leftrightarrow} \)

\( \mathcal{P} \): set of \( S \)-paths

\( \lambda : \mathcal{P} \to \mathbb{R}_+ \): flow-value function satisfying capacity constraint

\[
\sum\{\lambda(P) \mid P \in \mathcal{P} : e \in P\} \leq c(e) \quad (e \in EG).
\]
Multiflow feasibility problem

\(H\): demand graph with \(VH = S\)

\(q : EH \rightarrow \mathbb{R}_+\): demand function on edges \(EH\).

Find a multiflow \(f = (\mathcal{P}, \lambda)\) satisfying demand requirement

\[
\sum\{\lambda(P) | P \in \mathcal{P} : P \text{ is } (s,t)\text{-path}\} = q(st) \quad (st \in EH),
\]

or establish that no such a multiflow exists.

We are interested in behavior of multiflows for a fixed \(H\) and arbitrary \(G, c, q\).
\[ H = K_2: \text{ single commodity flows} \]

**Theorem** [Ford-Fulkerson 54]

\[ c, q \text{ integral, feasible } \implies \exists \text{ integral solution.} \]

\[ H = K_2 + K_2: \text{ 2-commodity flows} \]

**Theorem** [Hu 63]

\[ c, q \text{ integral, feasible } \implies \exists \text{ half-integral solution.} \]

\[ H = K_2 + K_2 + \cdots + K_2: \text{ } k\text{-commodity flows} \]

\[ c, q \text{ integral, feasible } \implies \exists 1/p\text{-integral solution } (p \leq k) \]

(Jewell 67, Seymour 81)
Theorem [Lomonosov 85]
There is no integer \( k > 0 \) such that every feasible 3-commodity flow problem with integer capacity and demand has a \( 1/k \)-integral solution.

Fractionality
\[
\text{frac}(H) := \text{the least positive integer } k \text{ with the property: }
\forall c, q \text{ integral, feasible } \Rightarrow \exists 1/k\text{-integral solution.}
\]

Problem [Karzanov 89,90]
\textit{Classify demand graphs } H \textit{ with } \text{frac}(H) < +\infty.\]

Remark \( H \supseteq K_2 + K_2 + K_2 \Rightarrow \text{frac}(H) = +\infty.\]
Graphs without $K_2 + K_2 + K_2$

(I) $K_4$, $C_5$, or star + star,

(II) $K_5$ or star + $K_3$,

(III) $K_3 + K_3$. 
(I) $H = K_4$, $C_5$, or star + star,

**Theorem** [Rothschild & Winston 66, Seymour 80, Lomonosov 76,85]
$c, q$ Eulerian, feasible $\Rightarrow \exists$ integral solution $(\to \frac{\text{frac}(H)}{2})$.

**Combinatorial feasibility condition** [Papernov 76]
$c, q$ feasible $\iff$ cut condition

$$\langle c, \delta_X \rangle_{EG} \geq \langle q, \delta_X \rangle_{EH} \quad (\forall X \subseteq VG),$$

where $\delta_X$ is the cut metric of $X$.

$$
\begin{array}{c|cc}
\delta_X & X & \bar{X} \\
\hline
X & 0 & 1 \\
\bar{X} & 1 & 0 \\
\end{array}
$$

\( (G, c) \) \quad \( (H, q) \)
(II) $H = K_5$ or star $+ K_3$

**Theorem** [Karzanov 87]
$c, q$ Eulerian, feasible $\Rightarrow \exists$ integral solution ($\rightarrow \frac{H}{2} = 2$).

**Combinatorial feasibility condition** [Karzanov 87]
$c, q$ feasible $\Leftrightarrow K_{2,3}$-metric condition

$$\langle c, d \rangle_{EG} \geq \langle q, d \rangle_{EH} \quad (\forall K_{2,3}$-metric $d$ on $VG$).

$K_{2,3}$-metric $d \overset{\text{def}}{\iff} d = d_{K_{2,3}}(\phi(\cdot), \phi(\cdot))$ for $\exists \phi : VG \rightarrow VK_{2,3}$

$$
\begin{array}{c|ccccc}
 & S & T & U_1 & U_2 & U_3 \\
\hline
S & 0 & 2 & 1 & 1 & 1 \\
T & 2 & 0 & 1 & 1 & 1 \\
U_1 & 1 & 1 & 0 & 2 & 2 \\
U_2 & 1 & 1 & 2 & 0 & 2 \\
U_3 & 1 & 1 & 2 & 2 & 0 \\
\end{array}
$$
(III) $H = K_3 + K_3$

**Remark** $\exists c, q$ integral, feasible  
$\Rightarrow$ no integral, no half-integral, $\exists$ quarter-integral solution.  
$\rightarrow \frac{c, q}{\Gamma_3,3} \geq 4$.

**Combinatorial feasibility condition** [Karzanov 89]

$c, q$ feasible $\Leftrightarrow \Gamma_{3,3}$-metric condition

**Conjecture** [Karzanov 90, ICM, Kyoto]

1. $\frac{c, q}{\Gamma_3,3} < +\infty.$

2. $c, q$ Eulerian, feasible $\Rightarrow \exists$ half-integral solution ($\rightarrow \frac{c, q}{\Gamma_3,3} = 4$)

Cf. Problems 51, 52 in Schrijver's book *"Combinatorial Optimization"*
Main Theorem [H. 08]

\[ H = K_3 + K_3, \ c, q \ \text{Eulerian, feasible} \]
\[ \Rightarrow \exists \ 1/12\text{-integral solution.} \]

\[ \rightarrow \text{the complete classification of} \]
\[ \text{demand graphs having finite fractionality} \]

Corollary \[ \frac{\text{frac}(H)}{2} < +\infty \iff H \not\supseteq K_2 + K_2 + K_2. \]

Corollary \[ \text{frac}(K_3 + K_3) \in \{4, 8, 12, 24\}. \]
Proof Sketch

1. Reduction to
   \( K_{3,3} \)-metric-weighted maximum multiflow problem

2. A combinatorial dual problem

3. Its optimality criterion

4. Fractional splitting-off with potential update
$K_{3,3}$-metric weighted maximum multiflow problem

$(G, c)$: an undirected graph with edge-capacity

$S \subseteq VG$: 6-element terminal set with $S = VK_{3,3}$

Max. $\sum_{P \in \mathcal{P}} d_{K_{3,3}}(s_P, t_P)\lambda(P)$

s. t. $f = (\mathcal{P}, \lambda):$ multiflow for $(G, c; S)$

Remark no integral optimum even if $(G, c)$ inner Eulerian.
Remark
\[ 1/k \text{-integral optimum in inner Eulerian } K_{3,3}\text{-max problem} \]
\[ \implies \]
\[ 1/k \text{-integral solution in Eulerian } K_{3} + K_{3}\text{-feasibility problem} \]

Theorem [H. 08]
\[ 1/12\text{-integral optimum in every inner Eulerian } K_{3,3}\text{-max problem}. \]
Splitting-off for multiflows
[Rothschild-Winston 66, Lovász 76, Seymour 80, Karzanov 87]

• Very powerful for showing an integral optimum in Eulerian problems.
• How about showing a $1/k$-integral optimum ($k \geq 2$)?
• A naive fractional variant violates Eulerianess, and induction fails.
Three key ingredients

- Combinatorial dual problem [Karzanov 89, 98]
- Its optimality criterion [H. 08]
- Fractional splitting-off with potential update [H. 08]
Combinatorial duality relation [Karzanov 89, 98]

Max. \[ \sum_{P \in \mathcal{P}} d_{K_n,n}(s_P, t_P) \lambda(P) \quad \text{s.t.} \quad f = (\mathcal{P}, \lambda) \quad \text{for} \quad (G, c; S) \]

= Min. \[ \frac{1}{2} \sum_{xy \in \mathcal{E}} c(xy) d_{\Gamma_n,n}(\rho(x), \rho(y)) \]

s. t. \[ \rho : V_G \to V_{\Gamma_n,n}, \quad \rho|_S = \text{id}, \quad \leftarrow \text{potential} \]
Optimality criterion [H. 08]

\[ \rho^0 \]: a forward neighbor of \( \rho \) \( \overset{\text{def}}{\iff} \) in forward orientation \( \Gamma_{3,3} \),

\[ \rho'(x) \neq \rho(x) \implies \rho(x)\rho'(x) \in E\Gamma_{3,3} \text{ or } (\rho(x), \rho'(x)) = (\bullet, \bullet) \text{ or } (\bullet, \bullet) \]

\[ \rho' \]: a backward neighbor of \( \rho \) \( \overset{\text{def}}{\iff} \) in backward orientation ...

Proposition [H. 08]

\( \rho \) is not optimal \( \Rightarrow \) \( \exists \) neighbor \( \rho' \) of \( \rho \) having smaller obj. value.
Proof sketch (of duality relation and opt. criterion)

LP-dual to $K_{3,3}$-max problem

$$\text{Min. } \sum_{xy \in EG} c(xy)d(x,y)$$
$$\text{s.t. } d: \text{ metric on } VG, \quad d|_S = d_{K_{3,3}}$$

**Proposition** [Karzanov 98]

every minimal metric is embedded into $(T_{K_{3,3}}, l_1)$.

**Lemma**

$$d_{l_1} \circ \rho = \sum_i \lambda_i (d_{l_1} \circ \rho_i) \text{ for } \exists \rho_i \text{ with } \text{Im } \rho_i = \{\bullet, \circ\}$$

cf. **tight spans** (Isbell 64, Dress 84)
Fractional splitting-off
\[ \tau = (e, y, e') : \text{fork} \]

Splitting capacity: \[ \alpha(\tau) := \max\{0 \leq \alpha \leq 2 \mid \text{opt}(G, c) = \text{opt}(G^\tau, c - \alpha \chi_{e^\tau})\} \]

Corollary \((G, c)\): inner Eulerian, \(\rho\): optimal potential

\[ \alpha(\tau) = \min \left\{ \left. \frac{\langle c, \Gamma_{3,3} \circ \rho' \rangle - \langle c, \Gamma_{3,3} \circ \rho \rangle}{\Gamma_{3,3}(\rho'(y), \rho'(y^\tau))} \right| \rho' : \text{neighbor of } \rho \text{ with } \rho'(y) \neq \rho'(y^\tau) \right\} \]

\[ \in \left\{ 0, \frac{1}{2}, \frac{2}{3}, 1, \frac{4}{3}, \frac{3}{2}, 2 \right\}. \quad \langle c, \Gamma_{3,3} \circ \rho \rangle := \sum_{xy \in EG} c(xy)d_{\Gamma_{3,3}}(\rho(x), \rho(y)) \]

A neighbor attaining \(\alpha(\tau)\) is called a critical neighbor.
\( \rho: \) optimal potential

\[ (G, c) \]

\( \Gamma_{3,3} \)

\[ S_\rho = \{ x \in VG \mid \rho(x) = \bullet \text{ or } \circ \} \]
\[ M_\rho = \{ x \in VG \mid \rho(x) = \circ \} \]
\[ C_\rho = \{ x \in VG \mid \rho(x) = \bullet \} \]

**Proposition [H. 08]**

\((G, c)\) inner Eulerian, \( \rho \) optimal potential, \( y \in S_\rho \) inner node

\( \Rightarrow y \) has a splittable fork.

**Corollary** \( M_\rho \cup C_\rho = \emptyset \) \( \Rightarrow \exists \) integral optimum.

cf. splitting-off idea for 5-terminus flows \( H = K_5 \) in [Karzanov 87]
Splitting-off with Potential UPdate (SPUP)

ρ: an optimal potential

τ: a fork (unsplittable) at $M_\rho \cup C_\rho$

ρ': a critical neighbor of ρ w.r.t. τ

$$\alpha(\tau) = \min_{\rho'} \frac{\langle c, d_{G_3,3} \circ \rho' \rangle - \langle c, d_{G_3,3} \circ \rho \rangle}{d_{G_3,3}(\rho'(y), \rho'(y^\tau))} \in \{0, \frac{1}{2}, \frac{2}{3}, 1, \frac{4}{3}, \frac{3}{2}, 2\}$$

**SPUP:** $(G, c; \rho) \rightarrow (G^\tau, c - \alpha(\tau)\chi_{e^\tau}; \rho')$

This SPUP does not keep $(G, c)$ Eulerian, but $C_\rho$ decreases, and still $\alpha(\tau) \in \{0, \frac{1}{2}, \frac{2}{3}, 1, \frac{4}{3}, \frac{3}{2}, 2\}$ in the next forward SPUP
In forward SPUP, $C_\rho$ is nonincreasing, and $M_\rho$ is nonincreasing if $C_\rho = \emptyset$.

These observations suggest us a possibility to repeat forward SPUPs until $M_\rho \cup C_\rho = \emptyset$ with keeping $(G, kc)$ inner Eulerian for a fixed integer $k$. ($\Rightarrow \exists 1/k$-integral optimum)

**Proposition [H. 08]** We can do it for $k = 12$.

The proof is lengthy and complicated.
Concluding remarks

- We do not know whether 1/12 is tight.
- The bounded fractionality conjecture for $K_3 + K_3$ is a very special case of the conjecture (see Proceedings);

  For a terminal weight $\mu$, $\dim T_\mu \leq 2$ if and only if there exists $k > 0$ such that every Eulerian $\mu$-max problem has a $1/k$-integral optimum.

  $$T_\mu := \text{Minimal } \{ p \in \mathbb{R}^S \mid p(s) + p(t) \geq \mu(s,t) \, (s, t \in S) \}$$

- Recently we proved it for $k = 12$ [H. 09, in preparation]
- Half-integral $\Gamma_{3,3}$-metric packing [H. 07, Combinatorica, to appear]

Future works

- Improving the bound 1/12 ($\rightarrow 1/2$ ?).
- Augmenting path algorithms for multiflows ?
Appendix I
Proof sketch (based on Karzanov’s splitting-off idea for 5-terminus flows)

\[ (G, c; \rho) \rightarrow \rho(y) \rightarrow (G', c - \alpha(\tau) \chi_{e_\tau}; \rho') \]

\[ \Rightarrow \alpha(\tau) \in \{0, 1, 2\}. \]

Suppose \( \alpha(\tau) = 1 \).

Take an optimal flow \( f = (\mathcal{P}, \lambda) \) for \( (G'^{\tau}, c - \alpha(\tau) \chi_{e_\tau}; \rho') \).

Complementary slackness: both \( f = (\mathcal{P}, \lambda), \rho' \) optimal \( \Leftrightarrow \)

\[ \rho'(x) \neq \rho'(y) \Rightarrow xy \text{ is saturated by } f, \]

\[ P \in \mathcal{P} : \lambda(P) > 0 \Rightarrow \rho'(P) \text{ is geodesic in } \Gamma_{3,3}. \]
if $\epsilon = 0$
if $\epsilon > 0$

$f$ is also optimal for $(G^\tau', c - \chi_{e\tau'})$
unsaturation of $e^{\tau'} > 1$
$\alpha(\tau') > 1 \Rightarrow \alpha(\tau') = 2$
$\tau'$ is splittable

$y$
$\tau'$
$\tau''$
splittable!

$g \geq 1/2$
$g > 1/2$
Appendix II
\((G, c; \rho)\): \textit{restricted Eulerian} if \(c\) integer, and \(\forall y \in M_\rho \cup C_\rho\) has even degree.

**Lemma** \((G, c; \rho)\): \textit{restricted Eulerian}, \(\tau\): a fork, \(\rho'\): a critical neighbor. \(\rho'\) is forward \(\Rightarrow \alpha(\tau) \in \{0, \frac{1}{2}, \frac{2}{3}, 1, \frac{4}{3}, \frac{3}{2}, 2\}\).

**Proposition** \((G, c; \rho)\): \textit{restricted Eulerian}, \(y \in M_\rho\) (unsplittable),

1. \(\exists\) an optimal forward neighbor \(\rho'\) with \(\rho'(y) \in S_\rho\), or

2. \(\exists\) a fork \(\tau\) s.t. a critical neighbor \(\rho'\) is forward

   \((\rightarrow \rho'(y), \rho'(y^\top) \in S_\rho, \alpha(\tau) = 1, \text{SPUP keeps} (G, c; \rho) \text{ restrict Eulerian})\)

**Corollary**
\((G, c; \rho)\) \textit{restricted Eulerian} with \(C_\rho = \emptyset \Rightarrow \exists\) half-integral optimum.
Starting from inner Eulerian \((G, c)\) with all inner node having degree four.

**Proposition** We can apply forward SPUPs to all degree four nodes in \(C_\rho\) with keeping \((G, 6c; \rho)\) restricted Eulerian.

**The ring condition:**

the subgraph of \(G\) induced by \(C_\rho\) consists of paths and cycles.

**Proposition** \((G, c; \rho)\) restricted Eulerian and the ring condition

\[ \Rightarrow \exists \text{ half-integral optimum.} \]