Half-integrality of node-capacitated multiflows and tree-shaped facility locations on trees

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Abstract

In this paper, we establish a novel duality relationship between node-capacitated multiflows and tree-shaped facility locations. We prove that the maximum value of a tree-distance-weighted maximum node-capacitated multiflow problem is equal to the minimum value of the problem of locating subtrees in a tree, and the maximum is attained by a half-integral multiflow. Utilizing this duality, we show that a half-integral optimal multiflow and an optimal location can be found in strongly polynomial time. These extend previously known results in the maximum free multiflow problems. We also show that the set of tree-distance weights is the only class having bounded fractionality in maximum node-capacitated multiflow problems.

1 Introduction

A node-capacitated *network* (V, E, S, b, c) consists of an undirected graph (V, E), a specified node subset $S \subseteq V$, called a *terminal set*, a nonnegative integral node-capacity $b : V \to \mathbf{Z}_+$, and a nonnegative integral edge-capacity $c : E \to \mathbf{Z}_+$. A path connecting distinct terminals is called an *S-path*. A *multiflow* (*multicommodity flow*) is a pair $(\mathcal{P}, \lambda)$ of a set $\mathcal{P}$ of S-paths and a nonnegative flow-value function $\lambda : \mathcal{P} \to \mathbf{R}_+$ satisfying the capacity constraint:

$$\begin{aligned} \sum\{\lambda(P) \mid P \in \mathcal{P}, x \in VP\} &\leq b(x) \quad (x \in V), \\ \sum\{\lambda(P) \mid P \in \mathcal{P}, e \in EP\} &\leq c(e) \quad (e \in E), \end{aligned}$$

where VP and EP denote the sets of nodes and edges in P, respectively. Let $\mu : \binom{S}{2} \to \mathbf{Q}_+$ be a nonnegative rational-valued function defined on the set $\binom{S}{2}$ of terminal pairs. For a multiflow $f = (\mathcal{P}, \lambda)$, the total flow-value $\mathrm{val}(\mu, f)$ with respect to μ is defined by $\sum_{P \in \mathcal{P}} \mu(s_P, t_P)\lambda(P)$, where s_P and t_P denote the ends of P. We consider the following problem:

(1.1) Maximize $\mathrm{val}(\mu, f)$ over all multiflows f in (V, E, S, b, c).

Example 1. Consider the case $S = \{s, t\}$ and $\mu(s, t) = 1$. This is a maximum flow problem. By the max-flow min-cut theorem or a version of Menger's theorem, the maximum flow value is equal to the minimum cut value, and there exists an *integral* maximum flow in (1.1).

Example 2. Consider the case where $|S| \geq 3$ and $\mu(s,t) = 1$ for all terminal pairs s, t. Then (1.1) is the fractional S-path packing problem, or sometimes called the maximum *free multiflow* problem. In this case, the integrality theorem does not hold. In the edge-only-capacitated case ($b \to +\infty$), Lovász [27] and Cherkassky [4] proved the existence of a *half-integral* optimal multiflow, and a combinatorial min-max formula. Also, in the node-capacitated case, a similar property holds. Garg, Vazirani, and Yannakakis [7] proved the dual half-integrality; also see [42, Section 20]. Pap [33, 34] proved the primal half-integrality and gave a strongly polynomial time algorithm to find a half-integral optimal multiflow; also see a further algorithmic development [2]. Note that this half-integrality can also be derived from Mader's disjoint S-paths theorem [28, 29].

The main contribution of this paper is to extend this half-integrality result to more general weights μ related to *trees*, with establishing a combinatorial min-max relation by a *tree-shaped facility location* on the tree associated with μ. A rational terminal weight $\mu : \binom{S}{2} \to \mathbf{Q}_+$ is called a *tree distance* if there exist a positive rational γ (scaling factor), a tree $\Gamma = (V\Gamma, E\Gamma)$, and a family $\{R_s \mid s \in S\}$ of its subtrees indexed by S such that

$$\mu(s,t) = \gamma d_\Gamma(R_s, R_t) \quad (s,t \in S).$$

Here subtree R_s is regarded as a node subset, and $d_\Gamma(R_s, R_t)$ denotes the shortest path distance between R_s and R_t. Namely $\mu(s,t)$ is represented as the distances between subtrees $\{R_s\}$ in tree Γ with uniform edge-length γ. We also say that $(\Gamma, \{R_s\}_{s \in S}; \gamma)$ *realizes* μ or $(\Gamma, \{R_s\}_{s \in S}; \gamma)$ *is a tree-realization of* μ. We particularly call μ a *tree metric* if each R_s is a single node. As far as we know, tree distances were first studied by [11] although tree metrics have been well studied in connection with the phylogenetic tree reconstruction in the biology [38]. Now suppose that μ is a tree distance realized by $(\Gamma, \{R_s\}_{s \in S}; \gamma)$. Let $\mathcal{F}\Gamma \subseteq 2^{V\Gamma}$ be a set of all subtrees in Γ. For a subtree F, the *diameter* $\mathrm{diam} F$ is defined by the maximum distance of nodes in F. Consider the following subtree location problem:

$$\begin{array}{lll} (1.2) \qquad & \text{Min.} & \gamma \sum_{y \in V} b(y) \mathrm{diam} F(y) + \gamma \sum_{xy \in E} c(xy) d_\Gamma(F(x), F(y)) \\ & \text{s.t.} & F : V \to \mathcal{F}\Gamma, \\ & & F(s) \cap R_s \neq \emptyset \quad (s \in S). \end{array}$$

This problem can be read as follows. We associate each x with a tree-shaped facility $F(x)$ in Γ. Each pair of facilities $F(x), F(y)$ interacts, and has a *communication cost* which is a nondecreasing function of the distance $d_\Gamma(F(x), F(y))$. So if the subtrees become large, then the communication cost becomes small. However this could be balanced by a *size cost* which is a nondecreasing function of diameter $\mathrm{diam} F(x)$. Then the problem is to find a subtree configuration of the smallest cost. We particularly call $F : V \to \mathcal{F}\Gamma$ a *subtree map*.

Our main theorem establishes the duality between (1.1) and (1.2) together with half-integrality property:

Theorem 1.1. *Let μ be a tree distance on S realized by $(\Gamma, \{R_s\}_{s \in S}; \gamma)$. For any network (V, E, S, b, c), the maximum value of* (1.1) *is equal to the minimum value of* (1.2). *Moreover there exists a half-integral optimal solution in* (1.1).

We also show that a half-integral optimal multiflow and an optimal subtree map can be found in strongly polynomial time.

A problem of locating several tree-shaped facilities in a tree network was studied by Tamir and Lowe [39] and Hakimi, Schmeichel, and Labbe [9], extending single tree-shaped facility location by Minieka [30]. In their model, subtrees are imposed to be disjoint, and the size cost (constraint) is the number of edges, instead of the diameter in our model. Many problem formulations (center, median, and etc) were shown to be NP-hard [9]. So the polynomial time solvability of (1.2) has own interest in the location theory.

Consider the case where the network is edge-only-capacitated ($b \to +\infty$). Then the diameter of each subtree $F(x)$ must be zero, i.e., $F(x)$ is a single point. Thus (1.2) reduces to a point-location problem on a tree:

$$(1.3) \qquad \begin{aligned} \text{Min.} \quad & \gamma \sum_{xy \in E} c(xy) d_{\Gamma}(\rho(x), \rho(y)) \\ \text{s.t.} \quad & \rho : V \to V\Gamma, \\ & \rho(s) \in R_s \quad (s \in S). \end{aligned}$$

The location problem of this type has been well studied in the literature [40]. In the multiflow theory, the corresponding duality relationship has been discovered by Karzanov [25, 26], and further developed by the author [12, 14, 15], for more general weights beyond tree distances. So our main theorem can be regarded as a node-capacitated variation of the results.

In examples below, a network is supposed to be node-only-capacitated. In this case, for each edge $xy \in E$, two facilities $F(x)$ and $F(y)$ must intersect, i.e., $d_{\Gamma}(F(x), F(y)) = 0$. Then (1.2) reduces to:

$$(1.4) \qquad \begin{aligned} \text{Min.} \quad & \gamma \sum_{y \in V} b(y) \mathrm{diam} F(y) \\ \text{s.t.} \quad & F : V \to \mathcal{F}\Gamma, \\ & F(x) \cap F(y) \neq \emptyset \quad (xy \in E), \\ & F(s) \cap R_s \neq \emptyset \quad (s \in S). \end{aligned}$$

Here a subtree map F with $F(x) \cap F(y) \neq \emptyset$ $(xy \in E)$ is identified with a family $\{V_p \mid p \in V\Gamma\}$ of nodes satisfying

(1) $V = \bigcup_{p \in V\Gamma} V_p$,

(2) for every edge $e = xy \in E$ there is $p \in V\Gamma$ with $x, y \in V_p$, and

(3) $V_p \cap V_q \subseteq V_r$ if r lies on the unique shortest path between p and q in Γ.

Indeed, consider $V_p := \{x \in V \mid p \in F(x)\}$ from F, and $F(x) := \{p \in V\Gamma \mid x \in V_p\}$ from $\{V_p\}_{p \in V\Gamma}$. A pair $(\Gamma, \{V_p\}_{p \in V\Gamma})$ is nothing but a *tree-decomposition* of (V, E) [36], a fundamental concept in the graph minor theory and the algorithmic graph theory. So the problem (1.4) is an optimization over tree-decompositions on a fixed tree Γ. Also this gives a unified interpretation of *combinatorial dual solutions* as follows. For simplicity, we assume that the node-capacity of each terminal s is sufficiently large ($b(s) \to +\infty$), and there is no edge joining terminals.

Example 3. Consider single flows; $S = \{s, t\}$ and $\mu(s, t) = 1$. Obviously μ is a tree metric realized by one edge $v_s v_t$ with $R_s = \{v_s\}$ and $R_t = \{v_t\}$ and $\gamma = 1$. Then (1.2) is given by

$$\text{(1.5)} \qquad \begin{aligned} \text{Min.} \quad & \sum_{y \in V \setminus \{s,t\}} b(y) \mathrm{diam} F(y) \\ \text{s.t.} \quad & F : V \to \{\{v_s\}, \{v_s, v_t\}, \{v_t\}\}, \\ & F(x) \cap F(y) \neq \emptyset \quad (xy \in E), \\ & (F(s), F(t)) = (\{v_s\}, \{v_t\}). \end{aligned}$$

For a feasible solution F in (1.5), the inverse image of $\{v_s, v_t\}$ is an (s,t)-node cut since there is no edge between $F^{-1}(\{v_s\})$ and $F^{-1}(\{v_t\})$. Moreover the objective value is the sum of the node-capacity over $F^{-1}(\{v_s, v_t\})$. Hence the maximum flow value is equal to the minimum capacity of (s,t)-node cuts.

Example 4. Consider the free multiflow; $S = \{s_1, s_2, \ldots, s_k\}$ and $\mu(s_i, s_j) = 1$ for each i, j. Then μ is a tree metric realized by a star Γ of k leaves $v_1, v_2, \ldots, v_k$ with center v_0, edge-length $\gamma = 1/2$, and $R_{s_i} = \{v_i\}$ for $i = 1, 2, \ldots, k$. Then (1.2) is given by

$$\text{(1.6)} \qquad \begin{aligned} \text{Min.} \quad & \frac{1}{2} \sum_{y \in V \setminus S} b(y) \mathrm{diam} F(y) \\ \text{s.t.} \quad & F(x) = \{v_i\}, \{v_j, v_0\}, \text{or } \{v_0, v_1, v_2, \ldots, v_k\} \quad (x \in V), \\ & F(x) \cap F(y) \neq \emptyset \quad (xy \in E), \\ & F(s_j) = \{v_j\} \quad (j = 1, 2, \ldots, k). \end{aligned}$$

This problem can also be represented as

$$\text{(1.7)} \qquad \begin{aligned} \text{Min.} \quad & b(U_0) + \frac{1}{2} \sum_{i=1}^{k} b(\mathrm{Bd}\, U_i) \\ \text{s.t.} \quad & \text{disjoint node subsets } U_0, U_1, U_2, \ldots, U_k, \\ & s_i \in U_i \quad (i = 1, 2, \ldots, k), \end{aligned}$$

where $\mathrm{Bd}\, U_i$ is the set of nodes in U_i incident to $V \setminus (U_i \cup U_0)$. This min-max formula coincides with one given in [42, Section 19.3], and can also be derived from Mader's S-path theorem [29]; see [34]. The equivalence between (1.6) and (1.7) can be seen as follows. For $U_0, U_1, U_2, \ldots, U_k$ feasible to (1.7), define subtree map $F : V \to \mathcal{F}\Gamma$ by

$$F(x) = \begin{cases} \{v_0, v_1, v_2, \ldots, v_k\} & \text{if } x \in U_0, \\ \{v_0, v_i\} & \text{if } x \in \mathrm{Bd}\, U_i, \\ \{v_i\} & \text{if } x \in U_i \setminus \mathrm{Bd}\, U_i, \\ \{v_0\} & \text{otherwise.} \end{cases}$$

Then F is feasible to (1.6) with the same objective value. Conversely, for a feasible subtree map F in (1.6), let $U_0 = \{x \in V \mid F(x) = \{v_0, v_1, v_2, \ldots, v_k\}\}$ and $U_i = \{x \in V \mid F(x) = \{v_i\} \text{ or } \{v_0, v_i\}\}$. Then $U_0, U_1, \ldots, U_k$ are feasible to (1.7) and does not increase the objective value by $\mathrm{Bd}\, U_i \subseteq F^{-1}(\{v_0, v_i\})$.

The rest of the paper is organized as follows. In Section 2, we give some preliminary arguments fractional b-matchings, metric-trees, and tree distances. In Section 3, we prove the former part of Theorem 1.1. We reduce (1.1) to an edge-only-capacitated problem with clique constraints (Section 3.1). By the LP-duality, we get a monotone optimization over metrics as a dual. Next applying the Helly argument, which was earlier developed by the author [14], we transform this metric-dual into a kind of continuous tree-location problem (Section 3.2). Then we get a natural continuous relaxation of (1.2), and finally show that the optimum is attained by a discrete one (Section 3.3). In Section 3.4, we present a geometric duality framework for (1.1) by the *tight span* T_μ of μ, which extends one in [12] to a node-capacitated version. Theorem 3.8 says that the dual of (1.1) becomes a ball location problem on T_μ. In particular, the case $\dim T_\mu \leq 1$ is exactly when μ is a tree distance [11]. This gives an insight on the node-capacitated multiflow duality.

In Section 4, we prove a stronger half-integral assertion: there exists a half-integral optimal multiflow of *minimum total cost.* This extends a classical result by Karzanov on minimum cost free multiflows [20], and its node-capacitated extension by Pap [33, 34] and Babenko-Karzanov [3]. Again the LP-dual of the minimum cost problem reduces to a *convex-cost* continuous tree-shaped facility location. Our proof is a combination of this subtree representation and the ideas of Karzanov [23] (construction of an optimal multiflow from a dual optimum) and of Pap [34] (use of fractional b-matchings). By the optimality criterion with an optimal subtree map F^*, we can construct a half-integral fractional b-matching ζ^*, which coincides with the flow-support of some optimal multiflow. From ζ^*, we can recover a half-integral optimum f^*. These constructions can be done in a strongly polynomial time with the help of Tardos' method [41]. A design of a combinatorial strongly polynomial algorithm is still open. In Section 5, we prove that the set of tree distances is the only class admitting the half-integrality theorem, more strongly, if μ is not a tree distance, then there is no positive integer k such that (1.1) has a $1/k$-integral optimal solution for every network (Theorem 5.1). This completes the fractionality problem in the maximum node-capacitated multiflow problem. Recently the edge-capacitated version of this problem was solved in [14, 15]. Interestingly, the both cases are characterized by the dimension of the tight span T_μ (see Section 3.4):

- The edge-capacitated fractionality of μ is finite $\Leftrightarrow$ $\dim T_\mu \leq 2$ [15].
- The node-capacitated fractionality of μ is finite $\Leftrightarrow$ $\dim T_\mu \leq 1$ (a consequence of this paper).

In Section 6, we give some concluding remarks.

Notation. Let $\mathbf{R}$, $\mathbf{Q}$, and $\mathbf{Z}$ denote the sets of reals, rationals, and integers, respectively and let $\mathbf{R}_+$, $\mathbf{Q}_+$, and $\mathbf{Z}_+$ denote the sets of nonnegative reals, rationals, and integers, respectively. For a function b on a set V and $A \subseteq V$, let $b(A)$ denote the sum of $b(s)$ over $s \in A$. Throughout the paper, a graph $G = (V, E)$ is an undirected graph without parallel edges and loops. For a node $x \in V$, the set of all edges incident to x is denoted by δx. By a path we mean a simple path, and denote it by a chain of nodes, such as $P = (x_1, x_2, \ldots, x_m)$. We often do not distinguish $\{x\}$ and x.

A *(semi)metric* d on a set X is a function $d : X \times X \to \mathbf{R}$ satisfying $d(x,y) = d(y,x) \geq d(x,x) = 0$ and the triangle inequality $d(x,y) + d(y,z) \geq d(x,z)$ for $x, y, z \in X$. A pair (X, d) is called a *metric space.* For two subsets $A, B \subseteq X$, the distance $d(A, B)$ between A and B is defined by $\inf_{x \in A, y \in B} d(x,y)$. For a subset $R \in X$ and nonnegative $\epsilon \geq 0$, let $B(R, r)$ denote the set of points $y \in X$ with $d(R, y) \leq r$. In particular, if R consists of a single point p, then $B(p, r)$ is a ball with center p and radius r. A map $\rho : V \to X$ induces metric $(d \circ \rho)$ on V by $d(\rho(x), \rho(y))$ $(x, y \in V)$. The shortest path metric of a graph Γ (with unit length) is denoted by d_Γ. In the case where V is the vertex set of a graph (V, E), we regard metric d as an edge-length $d : E \to \mathbf{R}_+$ by $d(e) := d(x, y)$ for $e = xy \in E$.

2 Preliminaries

Fractional b-matching. We note a basic fact of fractional b-matchings. Let $G = (V, E)$ be a graph. Let $\underline{b}, \overline{b} : V \to \mathbf{R}_+$ be a lower- and a upper-bound functions on V. A *fractional b-matching* is a nonnegative function $\zeta : E \to \mathbf{R}_+$ satisfying

$$\underline{b}(y) \leq \zeta(\delta y) \leq \overline{b}(y) \quad (y \in V).$$

Let $P(\underline{b}, \overline{b})$ be the polyhedron formed by all fractional b-matchings.

Lemma 2.1. *Suppose that both $\underline{b}$ and $\overline{b}$ are integer-valued. Then $P(\underline{b}, \overline{b})$ is half-integral. Moreover, for each extreme point ζ^* and each node $y \in V$, $\zeta^*(\delta y)$ is integral.*

The latter property, noted by [34], plays a key role in constructing a half-integral multiflow f^* from a flow-support ζ^*. We give a proof for completeness.

Proof. As is well known, any fractional b-matching can be obtained by a circulation in a directed network constructed from $(V, E, \underline{b}, \overline{b})$ as follows. For each node $x \in V$, consider two nodes x^+, x^- with directed edge x^+x^- from x^+ to x^- of lower capacity $\underline{b}(x)$ and upper capacity $\overline{b}(x)$. For each edge $xy \in V$, consider two directed edges y^-x^+ and x^-y^+ of lower capacity 0 and upper capacity $+\infty$. Consider a circulation ϕ in the resulting directed network. Let $\zeta : E \to \mathbf{R}$ be defined by $\zeta(xy) := (\phi(x^-y^+) + \phi(y^-x^+))/2$ for $xy \in E$. Then ζ is a fractional b-matching. Conversely, every fractional b-matching can be obtained in this way. So every extreme point ζ^* of $P(\underline{b}, \overline{b})$ is the image of an extreme point ϕ^* of the circulation polyhedron defined by integral capacity, Thus ϕ^* is integral, and ζ^* is half-integral. By construction, we have $\zeta^*(\delta x) = \phi^*(x^+x^-) \in \mathbf{Z}$. □

Metric-tree. In our argument, a *continuous relaxation* of a tree plays a crucial role. A *metric-tree* $\mathcal{T}$ is a metric space isometric to a 1-dimensional contractible complex endowed with the length metric. The metric function is denoted by $d_{\mathcal{T}}$. A closed connected subset of $\mathcal{T}$ is called a *subtree*. The set of subtrees is denoted by $\mathcal{F}\mathcal{T}$. For two points p, q, let $[p, q]$ denote the set of points r with $d_{\mathcal{T}}(p, q) = d_{\mathcal{T}}(p, r) + d_{\mathcal{T}}(r, q)$. A metric-tree of two vertices is particularly called a *segment*. For a weight $\mu : \binom{S}{2} \to \mathbf{R}_+$, a *continuous tree-realization* is a pair $(\mathcal{T}, \{R_s\}_{s \in S})$ of a metric-tree $\mathcal{T}$ and a family $\{R_s\}_{s \in S}$ of subtrees such that $\mu(s, t) = d_{\mathcal{T}}(R_s, R_t)$ for every $s, t \in S$. We also say that $(\mathcal{T}, \{R_s\}_{s \in S})$ realizes μ.

Let Γ be a tree (in the graph-theoretical sense). From Γ we can construct a metric-tree as follows. For each edge $e = uv \in E\Gamma$, consider segment F_e with two vertices $p_{e,u}, p_{e,v}$ and unit length $d_{F_e}(p_{e,u}, p_{e,v}) = 1$. Then consider disjoint union $\bigcup_{e \in E} F_e$, and for each node $u \in V\Gamma$, identify points $p_{e,u}$ for all edge e incident to u; the image of $p_{e,u}$ is denoted by p_u. The resulting metric-tree is denoted by $\bar{\Gamma}$. Then $(V\Gamma, d_\Gamma)$ isometrically embeds into $(\bar{\Gamma}, d_{\bar{\Gamma}})$ by $v \mapsto p_v$. So we can identify v and p_v, and regard as $V\Gamma \subseteq \bar{\Gamma}$.

On the complexity of tree distances. Here we briefly discuss the complexity of a realization $(\Gamma, \{R_s\}_{s \in S}; \gamma)$ of a tree distance μ. One can check in strongly polynomial time whether a given weight $\mu : \binom{S}{2} \to \mathbf{Q}_+$ is a tree distance [10, 11]; see also Theorem 3.9 in Section 3.4. If μ is a tree distance, then one can decompose μ into a nonnegative sum of cut distances for a laminar cut family in a strongly polynomial time [10, Section 4]; in this reference, a laminar family of cuts is called a *compatible family of partial cuts*. It is known that the number of cuts is bounded by $O(|S|^2)$ [10, Remark 4.15]. From this decomposition, one can obtain a tree $\tilde{\Gamma} = (V\tilde{\Gamma}, E\tilde{\Gamma})$, a positive edge-length l, and a family $\{\tilde{R}_s\}_{s \in S}$ of subtrees with $\mu(s, t) = d_{\tilde{\Gamma}, l}(\tilde{R}_s, \tilde{R}_t)$ $(s, t \in S)$, where $d_{\tilde{\Gamma},l}$ denotes the shortest path distance with respect to edge-length l. Here there is a one-to-one correspondence between cuts and edges in $\tilde{\Gamma}$; this is a variation of a tree representation of a laminar family. So the number of edges is bounded by $O(|S|^2)$. Note that these procedures are not fast, compared with existing algorithms for (well-studied) tree metrics. By subdivision, one obtains a realization $(\Gamma, \{R_s\}_{s\in S}; \gamma)$ of μ. The size of Γ is not polynomially bounded in general. However one can easily manipulate Γ by polynomial expression $(\tilde{\Gamma}, \{\tilde{R}_s\}_{s\in S}, l)$.

Remark 2.2. If μ is an integer-valued tree distance, then μ can be realized by a tree of edge-length $\gamma = 1/2$. One can easily verify it; in a minimal expression $(\tilde{\Gamma}, \{\tilde{R}_s\}_{s\in S}, l)$, edge-length $l(e)$ can be represented as $l(e) = (\mu(s,t) + \mu(u,r) - \mu(s,u) - \mu(t,r))/2$ for (not necessarily distinct) $s, t, u, r \in S$.

3 Geometric and combinatorial duality

Here we prove the former part of Theorem 1.1. We consider a natural continuous relaxation of the subtree location problem (1.2). Let μ be a tree distance realized by $(\Gamma, \{R_s\}_{s\in S}; \gamma)$. By scaling, we assume $\gamma = 1$. Let $\bar{\Gamma}$ be the metric-tree corresponding to Γ, and let $\bar{R}_s$ denote the smallest subtree of $\bar{\Gamma}$ containing $R_s \subseteq V\Gamma \subseteq \bar{\Gamma}$. Then $(\bar{\Gamma}, \{\bar{R}_s\}_{s\in S})$ realizes μ. Given a network (V, E, S, b, c), consider the following continuous relaxation of (1.2):

$$(3.1) \qquad \begin{aligned} \text{Min.} \quad & \sum_{y\in V} b(y)\mathrm{diam} F(y) + \sum_{xy\in E} c(xy) d_{\bar{\Gamma}}(F(x), F(y)) \\ \text{s.t.} \quad & F : V \to \mathcal{F}\bar{\Gamma}, \\ & F(s) \cap \bar{R}_s \neq \emptyset \quad (s \in S). \end{aligned}$$

The goal of this section is to prove the following, that implies the former part of the main theorem.

Proposition 3.1. *The optimal values of three problems* (1.1), (1.2), *and* (3.1) *are the same.*

Actually (3.1) gives an upper bound of (1.1) since

$$(3.2) \qquad \begin{aligned} \sum_{P\in\mathcal{P}} \mu(s_P, t_P)\lambda(P) &= \sum_{P\in\mathcal{P}} d_{\bar{\Gamma}}(\bar{R}_{s_P}, \bar{R}_{t_P})\lambda(P) \\ &\leq \sum_{P\in\mathcal{P}} \lambda(P) \left\{ \sum_{y\in VP} \mathrm{diam} F(y) + \sum_{xy\in EP} d_{\bar{\Gamma}}(F(x), F(y)) \right\} \\ &\leq \sum_{y\in V} b(y)\mathrm{diam} F(y) + \sum_{xy\in E} c(xy) d_{\bar{\Gamma}}(F(x), F(y)). \end{aligned}$$

3.1 A reduction to the edge-capacitated problem

In the first step, we reduce our problem (1.1) to an edge-only-capacitated problem with *clique constraints*. This technique was inspired by a discussion with Gyula Pap. Let (V, E, S, b, c) be a network with terminal weight μ. Let us construct a new network as follows. For each edge $e = xy$, subdivide e into three edge $xx_e, x_e y_e, y_e x$. Next, for each inner node x, replace the star of neighbors $\{xx_e\}_{e\in\delta x}$ by the clique on $\{x_e\}_{e\in\delta x}$. For each terminal s, replace the star of neighbors by the clique on $\{s_e\}_{e\in\delta s} \cup \{s\}$; see Figure 1.

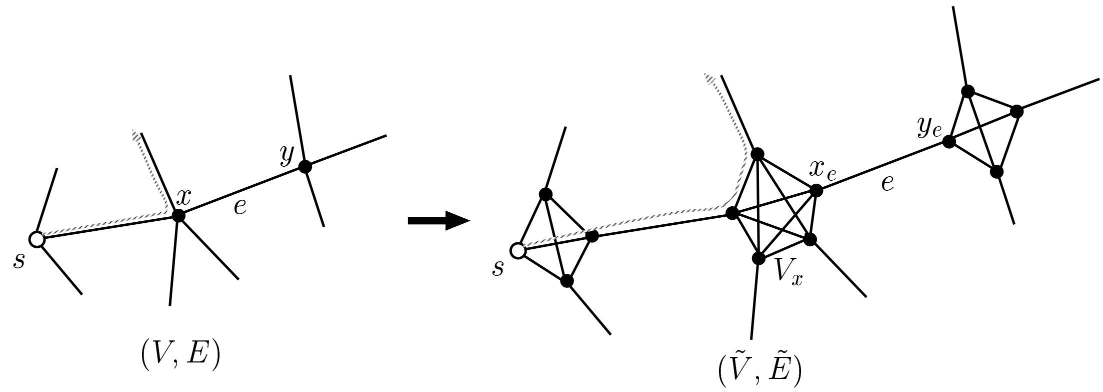

Figure 1: Construction of the extended graph

Let $(\tilde{V}, \tilde{E})$ be the resulting graph (called the *extended graph*). For $x \in V$, let V_x denote the set of nodes of the clique corresponding to x, and let E_x denote the set of edges in the clique of x. By the correspondence $E \ni e = xy \mapsto x_e y_e$, we regard E as $E \subseteq \tilde{E}$. Thus $\tilde{V} = \bigcup_{x \in V} V_x$ and $\tilde{E} = E \cup \bigcup_{x \in V} E_x$ (the both are disjoint union). Let $\mathit{\Pi}$ denote the set of all S-paths in $(\tilde{V}, \tilde{E})$. Then consider the following problem:

$$\text{(3.3)} \qquad \begin{array}{rl} \text{Max.} & \displaystyle\sum_{P \in \mathit{\Pi}} \mu(s_P, t_P)\lambda(P) \\ \text{s.t.} & \zeta(e) \leq c(e) \quad (e \in E), \\ & \zeta(E_x) \leq b(x) \quad (x \in V), \\ & \displaystyle\sum_{P \in \mathit{\Pi}, e \in EP} \lambda(P) \leq \zeta(e) \quad (e \in \tilde{E}), \\ & \zeta : \tilde{E} \to \mathbf{R}_+, \ \lambda : \mathit{\Pi} \to \mathbf{R}_+. \end{array}$$

A meaning of this problem is as follows. λ is nothing but an edge-only-capacitated multiflow with respect to the edge-capacity variable ζ which satisfies the edge-capacity constraint and the clique constraint corresponding to the node-capacity constraint in the original graph. The problems (1.1) and (3.3) are equivalent. To see this, take any multiflow $(\mathcal{P}, \lambda)$ in (1.1). Then any S-path in (V, E) is uniquely extended to an S-path in $(\tilde{V}, \tilde{E})$. So we can regard $\mathcal{P}$ as $\mathcal{P} \subseteq \mathit{\Pi}$. Define ζ by $\zeta(e) = \sum_{P \in \mathcal{P}, e \in EP} \lambda(P)$ for $e \in \tilde{E}$. We get a feasible solution (λ, ζ) with the same objective value. Conversely, take a feasible solution (λ, ζ) of (3.3). Contract each clique V_x to x. We get a multiflow in (1.1) with the same objective value.

Dualize (3.3). Then we obtain the following LP-dual.

$$\text{(3.4)} \qquad \begin{array}{rl} \text{Min.} & \displaystyle\sum_{x \in V} b(x)u(x) + \sum_{e \in E} c(e)\hat{l}(e) \\ \text{s.t.} & \hat{l}(e) \geq l(e) \ (e \in E), \\ & u(x) \geq l(e) \quad (e \in E_x, x \in V), \\ & l(EP) \geq \mu(s_P, t_P) \ (P \in \mathit{\Pi}), \\ & \hat{l} : E \to \mathbf{R}_+, l : \tilde{E} \to \mathbf{R}_+, u : V \to \mathbf{R}_+. \end{array}$$

Since b and c are nonngegative, we can take $\hat{l}, u$ as $\hat{l}(e) = l(e)$ for $e \in E$ and $u(x) = \max_{e \in E_x}\{l(e)\}$. Furthermore consider the shortest-path metric d_l on $\tilde{V}$ induced by l.

Then $d_l(e) \leq l(e)$, $d_l(s,t) \geq \mu(s,t)$ and hence we can replace l by metric d_l; this technique is standard in the multiflow theory. Hence the optimal value of (3.4) coincides with the following metric optimization:

$$\text{(3.5)} \qquad \begin{array}{ll} \text{Min.} & \displaystyle\sum_{x \in V} b(x) \max_{e \in E_x}\{d(e)\} + \sum_{e \in E} c(e) d(e) \\ \text{s.t.} & d : \text{metric on } \tilde{V}, \\ & d(s,t) \geq \mu(s,t) \quad (s,t \in S). \end{array}$$

An important point here is that LP (3.5) has $O(|\tilde{V}|^3) = O(|V|^6)$ number of inequalities with $\{-1,0,1\}$ coefficient. So it can be solved by Tardos' method [41] in strongly polynomial time.

3.2 Embedding into a tree

The second step is to derive a geometric dual problem (3.1) directly from (3.5). A monotone optimization over metrics d with $d(s,t) \geq \mu(s,t)$ has been recently well-studied by [12, 14]. There are two approaches to get (3.1): the first one is based on the Helly property [14], and the second one is based on the tight span [12]. Here we describe the first approach since it is self-contained. The second approach is summarized in Section 3.4.

A key lemma is the following, which is a special case of [14, Theorem 2.1 (1)],

Lemma 3.2. *Suppose that μ is realized by $(\Gamma, \{R_s\}_{s \in S}; 1)$. For every metric d on $\tilde{V}$ feasible to (3.5), there exists a map $\rho : \tilde{V} \to \bar{\Gamma}$ such that $\rho(s) \in \bar{R}_s$ for $s \in S$ and $(d_{\bar{\Gamma}} \circ \rho)(x,y) \leq d(x,y)$ for $x, y \in \tilde{V}$.*

We give the proof for completeness. It is a simple application of the Helly property of subtrees in a tree; this idea comes from the classic paper [1]. Note that this proof constructs ρ in strongly polynomial time.

Proof. Let $\tilde{V} = \{x_1, x_2, \ldots, x_n\}$. Define a map $\rho : \tilde{V} \to \bar{\Gamma}$ recursively by $\rho(x_k)$ being an arbitrary point in

$$\text{(3.6)} \qquad \bigcap_{s \in S} B(\bar{R}_s, d(s, x_k)) \cap \bigcap_{i=1}^{k-1} B(\rho(x_i), d(x_i, x_k)) \quad (k = 1, 2, \ldots, n).$$

We show that the set (3.6) is nonempty for all k, which implies that ρ is well-defined and satisfies $\rho(s) \in \bar{R}_s$ and $d_{\bar{\Gamma}} \circ \rho \leq d$; indeed, $\rho(s) \in B(\bar{R}_s, d(s,s)) = \bar{R}_s$, and $\rho(x_j) \in B(\rho(x_i), d(x_i, x_j))$ $(i < j)$ implies $d_{\bar{\Gamma}}(\rho(x_i), \rho(x_j)) \leq d(x_i, x_j)$.

We use the induction on k. Suppose that (3.6) is nonempty for $i < k$. By the Helly property of subtrees, it suffices to show that each pairwise intersection is nonempty. Note that two subtrees $B(R, r)$ and $B(R', r')$ intersect if and only if $d_{\bar{\Gamma}}(R, R') \leq r + r'$. For $s, t \in S$, the nonemptiness of $B(\bar{R}_s, d(s, x_k)) \cap B(\bar{R}_t, d(t, x_k))$ follows from $d(s, x_k) + d(x_k, t) \geq d(s,t) \geq \mu(s,t) = d_{\bar{\Gamma}}(\bar{R}_s, \bar{R}_t)$. For $s \in S$ and $i < k$, the nonemptiness of $B(\bar{R}_s, d(s, x_k)) \cap B(\rho(x_i), d(x_i, x_k))$ follows from $d(s, x_k) + d(x_k, x_i) \geq d(s, x_i) \geq d_{\bar{\Gamma}}(\bar{R}_s, \rho(x_i))$, where the last inequality follows from $\rho(x_i) \in B(\bar{R}_s, d(s, x_i))$ for $i < k$ (by induction). For $1 \leq i < j < k$, the nonemptiness of $B(\rho(x_i), d(x_i, x_k)) \cap B(\rho(x_j), d(x_j, x_k))$ follows from $d(x_i, x_k) + d(x_k, x_j) \geq d(x_i, x_j) \geq d_{\bar{\Gamma}}(\rho(x_i), \rho(x_j))$. Thus we are done. □

Thus (3.5) is further transformed into the following tree location problem:

$$\text{(3.7)} \qquad \begin{array}{ll} \text{Min.} & \sum_{x \in V} b(x) \max_{e \in E_x} \{(d_{\bar{\Gamma}} \circ \rho)(e)\} + \sum_{e \in E} c(e)(d_{\bar{\Gamma}} \circ \rho)(e) \\ \text{s.t.} & \rho : \tilde{V} \to \bar{\Gamma}, \\ & \rho(s) \in \bar{R}_s \quad (s \in S). \end{array}$$

For each $x \in V$, define $F(x)$ by the smallest subtree containing $\rho(y)$ for $y \in V_x$. Then we get a subtree map $F : V \to \mathcal{F}\bar{\Gamma}$ with the properties that $F(s) \cap \bar{R}_s \neq \emptyset$ $(s \in S)$, $d_{\bar{\Gamma}}(F(x), F(y)) \leq (d_{\bar{\Gamma}} \circ \rho)(e)$ for each $e = xy \in E$, and $\mathrm{diam} F(x) = \max_{e \in E_x} \{(d_{\bar{\Gamma}} \circ \rho)(e)\}$ for $x \in V$. Thus we have the part of Proposition 3.1; the optimal values of (1.1) and (3.1) are the same.

In fact the optimum is attained by a map F so that each $F(x)$ is a ball with radius $\mathrm{diam} F(x)/2$. In the argument above, consider $\bigcap_{u \in V_x} B(\rho(u), g/2)$ for $g := \max_{e \in E_x} \{(d_{\bar{\Gamma}} \circ \rho)(e)\}$. By the Helly property of subtrees, this intersection is nonempty and consists of a single point p. Let $p_x := p$ and $r_x := g/2$. Then $\rho(u) \subseteq B(p_x, r_x)$ for $u \in V_x$. Thus we also obtain the following expression of (3.1):

$$\text{(3.8)} \qquad \begin{array}{ll} \text{Min.} & \sum_{x \in V} 2b(x)r(x) + \sum_{e = xy \in E} c(e) d_{\bar{\Gamma}}(B(\rho(x), r(x)), B(\rho(y), r(y))) \\ \text{s.t.} & \rho : V \to \bar{\Gamma}, \ r : V \to \mathbf{R}_+, \\ & B(\rho(x), r(x)) \cap \bar{R}_s \neq \emptyset \quad (s \in S). \end{array}$$

3.3 Rounding

The final step is to *round* a map $F : V \to \mathcal{F}\bar{\Gamma}$ into $V \to \mathcal{F}\Gamma$, not increasing the objective value. For a positive integer k, a *$1/k$-integral point* of $\bar{\Gamma}$ is a point p such that $d_{\bar{\Gamma}}(p, u)$ is $1/k$-integral for some (arbitrary) $u \in V\Gamma$. The set of $1/k$-integral points is denoted by $V^k\Gamma$. In particular $V^1\Gamma = V\Gamma$. A subtree map F is said to be *$1/k$-integral* if each leaf of $F(x)$ for each $x \in V$ belongs to $V^k\Gamma$.

Let $F : V \to \mathcal{F}\bar{\Gamma}$ be a subtree map feasible to (3.1) obtained from a rational solution d in (3.5) according to the proof of Lemma 3.2. Then there is a positive integer k such that F is $1/k$-integral. Our goal is to show the existence of maps $F_i : V \to \mathcal{F}\bar{\Gamma}$ $(i = 1, 2, \ldots, k)$ with the properties:

(i) F_i is integral for $i = 1, 2, \ldots, k$.

(ii) $d_{\bar{\Gamma}}(F(x), F(y)) = \sum_{i=1}^{k} d_{\bar{\Gamma}}(F_i(x), F_i(y))/k$ for each $x, y \in V$.

(iii) $\mathrm{diam} F(x) = \sum_{i=1}^{k} \mathrm{diam} F_k(x)/k$ for each $x \in V$.

(iv) $F_i(s) \cap \bar{R}_s \neq \emptyset$ for $s \in S$ and $i = 1, 2, \ldots, k$.

If true, then we can always take an *integral* optimal solution F in (3.1), and get an optimal solution of (1.2) by restricting each $F(x)$ to $F(x) \cap V^1\Gamma$.

We fix an orientation of Γ with the property that each node is a source or a sink, i.e., we regard Γ as a bipartite graph of bipartition $\{A, B\}$, and orient each edge from A to B. Let Γ^k be the undirected graph on $V^k\Gamma$ with edge set $\{uv \mid d_{\bar{\Gamma}}(u, v) = 1/k\}$. Namely Γ^k is the k-subdivision of Γ. We can regard $F : V \to \mathcal{F}\bar{\Gamma}$ as $V \to \mathcal{F}\Gamma^k$. For an edge $uv \in E\Gamma$ with oriented as $\overrightarrow{uv}$, order the k subdivided edges $e_1, e_2, \ldots, e_k$ of uv from sink u to source v. We call e_i the *i-th edge* of uv. For $1 \leq i \leq k$, we define $F_i : V \to \mathcal{F}\Gamma$ as follows. Contract all edges except i-th edges in Γ^k. Then the resulting graph coincides with Γ. So we obtain a map $\varphi_i : V\Gamma^k \to V\Gamma$ by defining $\varphi_i(x)$ to be the contracted node. Obviously $\varphi_i \circ F(x)$ is a subtree in Γ, and the image of a leaf in $F(x)$ is a leaf in $\varphi_i \circ F(x)$. Let $F_i := \varphi_i \circ F$. We show that $F_1, F_2, \ldots, F_k$ satisfy properties (i-iv). (i) is obvious by construction. Also it is easy to see (iv). For (ii) and (iii), we first remark the following property:

(3.9) For a shortest path P connecting u and v in Γ^k, $\varphi_i(P)$ is also a shortest path connecting $\varphi(u)$ and $\varphi(v)$ in Γ.

We verify (ii). If $F(x) \cap F(y) \neq \emptyset$, then $F_i(x) \cap F_i(y) \neq \emptyset$, and thus (ii) is obvious. Therefore $F(x) \cap F(y) = \emptyset$. Take a unique shortest path P connecting $F(x)$ and $F(y)$. Then $\varphi_i(P)$ is also a unique shortest path connecting $F_i(x)$ and $F_i(y)$, and its length $d_\Gamma(F_i(x), F_i(y))$ is the number of the i-th edges in P. From this we have (ii).

Next we verify (iii). For $x \in V$, take a maximum farthest pair (u, u') of leaves in $F(x)$, i.e., $d_\Gamma(u, u') = \mathrm{diam} F(x)$. It suffices to show

$$d_\Gamma(\varphi_i(u), \varphi_i(u')) = \mathrm{diam} F_i(x) \quad (i = 1, 2, \ldots, k). \tag{3.10}$$

Namely $(\varphi_i(u), \varphi_i(u'))$ is again maximum farthest in $F_i(x)$. If true, then we can apply the argument above to a shortest path connecting u and u', and we get (iii). In the sequel, $d_{\bar{\Gamma}}$ is simply denoted by d. By (the argument before) (3.8), we may assume that $F(x)$ is a ball with center p and radius $\mathrm{diam} F(x)/2$. By taking a larger (redundant) tree-representation, we may further assume (for simplicity) that

(3.11) $d(p, u) = \mathrm{diam} F(x)/2$ for every leaf u of $F(x)$.

Take an edge $qq' \in E\Gamma$ such that segment $[q, q']$ contains p. We may assume $d(q, p) \leq d(q', p) < 1$. If $\mathrm{diam} F(x)/2 \leq d(q, p)$, then $F(x)$ is a path in $[q, q']$, and hence (3.10) is obvious. Suppose $\mathrm{diam} F(x)/2 > d(q, p)$. For a leaf v in $F(x)$, let v_* denote the nearest integral point in $F(x) \cap V^1\Gamma$. Then $d(\varphi_i(v), q) = d(v_*, q) + 1$ if $[v, v_*]$ contains an i-th edge and $d(\varphi_i(v), q) = d(v_*, q)$ otherwise. Let K' be the connected component of $\bar{\Gamma} \setminus p$ containing q', and let $K = F(x) \setminus K'$. By (3.11) the diameter is attained by any pair of leaves $(u, u') \in K \times K'$. Let $r = d(u_*, q)$, and let $r' = d(u'_*, q)$. Then we have

$$r' = r \text{ or } r' = r + 1.$$

Consider the image of leaves by φ_i. Recall the orientation and the ordering of edges; nodes at distance r (resp. r') from q are either all sources or all sinks. Therefore, for a pair of leaves $(u, u') \in K \times K'$, the distance $d(\varphi_i(u), \varphi_i(u'))(= d(\varphi_i(u), q) + d(q, \varphi_i(u')))$ is independent on the choice of (u, u').

Let (w, w') be an arbitrary pair of (distinct) leaves in $F(x)$. We show $d(\varphi_i(w), \varphi_i(w')) \leq d(\varphi_i(u), \varphi_i(u'))$, which implies (3.10). It suffices to consider the cases $(w, w') \in K \times K$ and $(w, w') \in K' \times K'$. If $(w, w') \in K' \times K'$ (necessarily $r' \geq 1$), then

$$\begin{aligned} d(\varphi_i(w), \varphi_i(w')) &\leq d(\varphi_i(w), q) + d(\varphi_i(w'), q) - 2 = 2d(\varphi_i(u'), q) - 2 \\ &\leq d(\varphi_i(u), q) + d(\varphi_i(u'), q) = d(\varphi_i(u), \varphi_i(u')), \end{aligned}$$

where we use $d(\varphi_i(u'), q) \leq r' + 1 \leq r + 2 \leq d(\varphi_i(u), q) + 2$.

Suppose $(w, w') \in K \times K$. Then we have

$$d(\varphi_i(w), \varphi_i(w')) \leq d(\varphi_i(w), q) + d(\varphi_i(w'), q) = 2d(\varphi_i(u), q).$$

Suppose $r = r' - 1$. Then $d(\varphi_i(u), q) \leq r + 1 = r' \leq d(\varphi_i(u'), q)$. Suppose $r = r'$. Then u_* and u'_* are either both sinks or both sources. By $d(u, u_*) \leq d(u', u'_*)$, if $[u, u_*]$

contains an i-th edge, then so does $[u', u'_*]$. Thus we have $d(\varphi_i(u), q) \leq d(\varphi_i(u'), q)$. Consequently we have $d(\varphi_i(w), \varphi_i(w')) \leq d(\varphi_i(u), \varphi_i(u'))$. Thus we get (iii), and the proof of Proposition 3.1 is complete.

Corollary 3.3. *There exists a strongly polynomial time algorithm to find an optimal subtree map in* (1.2).

Proof. Solving LP (3.5) by Tardos' method [41], we get an optimal rational metric d on $\tilde{V}$. Next following the proof of Lemma 3.2, we get an optimal map $\rho : \tilde{V} \to \bar{\Gamma}$. From this, we get an optimal subtree map $F : V \to \mathcal{F}\bar{\Gamma}$. We can take k so that F is $1/k$-integral. By the orientation of Γ (with bipartition $\{A, B\}$), decompose F into $F_1, F_2, \ldots, F_k$ as above. Each summand F_i is optimal. Although k may not be polynomial bounded, we can get F_1 (say) directly as follows. Indeed, $F_1(x)$ consists of vertices in $V\Gamma \cap F(x)$ and vertices $q \in B \setminus F(x)$ having edge $pq \in E\Gamma$ such that $p \in A \cap F(x)$ and segment $[p, q]$ meets $F(x)$ in the interior. These constructions can be done in strongly polynomial time. □

Remark 3.4. In fact, this rounding technique is the same as that in [12, 14, 25, 26] applied to a *folder complex* [5], a special cell-complex of folders. We briefly mention this connection. Fix a sufficiently large $K \in \mathbf{Z}_+$. Consider $|V|$ segments $\mathcal{Z}_x$ ($x \in V$) of length K. Consider the disjoint union $\bigcup_{x \in V} \mathcal{Z}_x$ and identify one of the ends of each segment. Then we get a metric-tree $\mathcal{T}$ of a star consisting of $|V|$ edges of length K. Let O denote the center of star $\mathcal{T}$. Next consider the product space $\bar{\Gamma} \times \mathcal{T}$. Then a pair (r, ρ) of maps feasible to (3.8) can be identified with a map $\varphi : V \to \bar{\Gamma} \times \mathcal{T}$ satisfying $\varphi(x) \in \bar{\Gamma} \times \mathcal{Z}_x$ for $x \in V \setminus S$ and $\varphi(s) \in \bar{R}_s \times \mathcal{Z}_s$ for $s \in S$. The objective is a function of the relative position of $\{\varphi(x)\}_{x \in V}$. For a positive integer $k \in \mathbf{Z}$, let $V^k\mathcal{T}$ denote the set of points $p \in \mathcal{T}$ with $d_{\mathcal{T}}(p, O) \in \mathbf{Z}/k$. Fix a point $(p^*, O) \in V\bar{\Gamma} \times V\mathcal{T}$. Consider the set of points $(p, q) \in V^{2k}\bar{\Gamma} \times V^{2k}\mathcal{T}$ with $d_{\bar{\Gamma}}(p, p^*) + d_{\mathcal{T}}(O, q) \in \mathbf{Z}/k$, denoted by $V^k(\bar{\Gamma} \times \mathcal{T})$; also $V^1(\bar{\Gamma} \times \mathcal{T})$ is denoted simply by $V(\bar{\Gamma} \times \mathcal{T})$. Join each pair of points $(p, q), (p', q') \in V^k(\bar{\Gamma} \times \mathcal{T})$ satisfying $d_{\bar{\Gamma}}(p, p') + d_{\mathcal{T}}(q, q') \in \{-1/k, 1/k\}$. Then we get the structure of a cell complex on $\bar{\Gamma} \times \mathcal{T}$ with vertex set $V^k(\bar{\Gamma} \times \mathcal{T})$, which in fact forms a folder complex; moreover it is orientable, and $\bar{\Gamma} \times \mathcal{Z}_x$ and $\bar{R}_s \times \mathcal{Z}_x$ are normal sets in the sense of [14]. See Figure 2; there are *folders* along $\bar{\Gamma} \times \{O\}$. As in [14], we can decompose $\varphi : V \to V^k(\bar{\Gamma} \times \mathcal{T})$ to $\varphi_i : V \to V(\bar{\Gamma} \times \mathcal{T})$, which induces the same decomposition of a subtree map as above. Since the objective function is different from that treated in [14], we need a more discussion to derive (i-iv) from this approach.

Such a folder structure arises ubiquitously from the combinatorial multiflow theory. Indeed, any 2-dimensional tight span (see below) has the structure of a folder complex, which plays a crucial role in the study of edge-only-capacitated multiflows [12, 14, 15, 26]. A folder complex obtained from the product of two trees (as above) also arises from the study of integer multiflows [17].

3.4 A geometric duality by tight span

Here we describe an alternative derivation of (3.1) according to the tight-span of μ. It has own interest since it gives a geometric dual problem sharpening LP-dual (3.5) for general μ (not necessarily a tree distance). This technique was earlier developed by [12] for the edge-capacitated case, based on pioneering works [25, 26] by Karzanov. See also [13] for an exposition.

Our argument here can be regarded as a natural extension of [12, Section 2] to a node-capacitated version. Let $\mu : \binom{S}{2} \to \mathbf{R}_+$ be a terminal weight. Let us define two

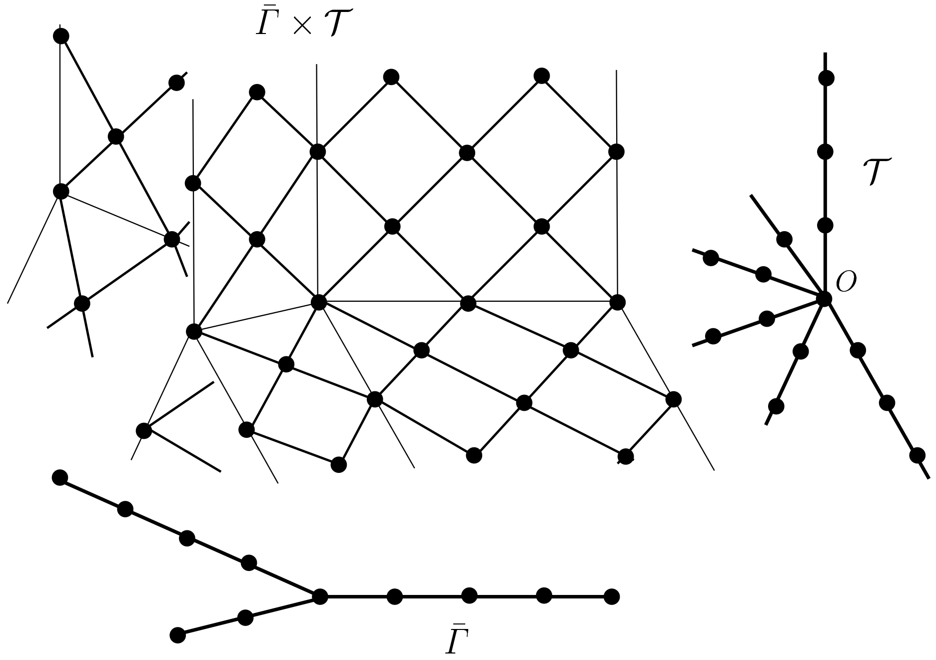

Figure 2: Product of two trees with a folder structure

polyhedral subsets P_μ, T_μ in $\mathbf{R}_+^S$ by

$$(3.12) \qquad \begin{array}{rcl} P_\mu & := & \{p \in \mathbf{R}_+^S \mid p(s) + p(t) \geq \mu(s,t) \ (s,t \in S)\}, \\ T_\mu & := & \text{the set of minimal points of } P_\mu. \end{array}$$

Here a point p in P_μ is said to be *minimal* if there is no point q in P_μ with $q \leq p$ and $q \neq p$. T_μ is called the *tight span* or the *injective hull* of μ, which was introduced independently by Isbell [19] and Dress [6] for metrics, and was considered by the author [11] for general weights. We endow the l_∞-metric d_∞ with P_μ, T_μ, i.e., $d_\infty(p,q) := \max_{s \in S} |p(s) - q(s)|$. For $s \in S$, we define subsets of P_μ, T_μ as

$$(3.13) \qquad \begin{array}{rcl} P_{\mu,s} & := & \{p \in P_\mu \mid p(s) = 0\}, \\ T_{\mu,s} & := & \{p \in T_\mu \mid p(s) = 0\}. \end{array}$$

Let (V, E, S, b, c) be a network. As in Section 3.1, construct the extended graph $(\tilde{V}, \tilde{E})$, and consider problems (3.3) and (3.5). Also consider the following location problem on P_μ:

$$(3.14) \qquad \begin{array}{ll} \text{Min.} & \displaystyle\sum_{x \in V} b(x) \max_{e \in E_x} (d_\infty \circ \rho)(e) + \sum_{e \in E} c(e)(d_\infty \circ \rho)(e) \\ \text{s.t.} & \rho : \tilde{V} \to P_\mu, \\ & \rho(s) \in P_{\mu,s} \quad (s \in S). \end{array}$$

This problem gives an upper bound of (3.5) since the induced metric $d_\infty \circ \rho$ on $\tilde{V}$ has the same objective and satisfies the feasibility: $(d_\infty \circ \rho)(s,t) \geq (\rho(s))(t) - (\rho(t))(t) = (\rho(s))(t) + (\rho(t))(t) \geq \mu(st)$ by $(\rho(t))(t) = 0$. Conversely we have:

Lemma 3.5 ([12]). *For every metric d on $\tilde{V}$ feasible to* (3.5)*, there exists a map ρ : $\tilde{V} \to P_\mu$ such that $(d_\infty \circ \rho)(x,y) \leq d(x,y)$ for $x, y \in \tilde{V}$, and $\rho(s) \in P_{\mu,s}$ for $s \in S$.*

Proof. Define ρ by $(\rho(x))(s) = d(x,s)$ $(x \in \tilde{V}, s \in S)$. Then $(\rho(x))(s) + (\rho(x))(t) = d(x,s) + d(x,t) \geq d(s,t) \geq \mu(s,t)$. So $\rho(x) \in P_\mu$ for each $x \in \tilde{V}$. Obviously $\rho(s) \in P_{\mu,s}$ since $(\rho(s))(s) = d(s,s) = 0$. Moreover $(d_\infty \circ \rho)(x,y) = \max_{s \in S} |d(x,s) - d(y,s)| \leq d(x,y)$ by the triangle inequality. □

Therefore the optimal values of (3.5) and (3.14) are the same. Furthermore we can replace P_μ (resp. $P_{\mu,s}$) by T_μ(resp. $T_{\mu,s}$) in (3.14) by Dress' nonexpansive retraction lemma:

Lemma 3.6 ([6, (1.9)])**.** *There exists a map $\varphi : P_\mu \to T_\mu$ such that $\varphi(p) \leq p$ for $p \in P_\mu$ and $(d_\infty \circ \varphi)(p,q) \leq d_\infty(p,q)$ for $p,q \in P_\mu$.*

See also [13] for a proof. Consider $\varphi \circ \rho$ for a feasible map ρ in (3.14), which keeps the objective and the feasibility.

Finally we derive a ball-expression similar to (3.8). Let us introduce the notion of the hyperconvexity by Aronzajn-Panitchpakdi [1]. A metric space (X,d) is said to be *hyperconvex* if for any collection of closed balls $\{B(x_i, r_i) \mid i \in I\}$, satisfying the condition that $d(x_i, x_j) \leq r_i + r_j$ for all $i, j \in I$, the intersection $\bigcap_{i \in I} B(x_i, r_i)$ is nonempty, i.e., (X,d) is geodesic and the collection of balls has the Helly property. Examples include metric-trees, l_∞-spaces, the l_1-plane, and so on. In particular (P_μ, d_∞) is hyperconvex. One can easily see that the image of a hyperconvex space by a nonexpansive retraction is also hyperconvex. So we get the following well-known fact in the literature; see [14, Section 4.2].

Lemma 3.7. *(T_μ, d_∞) is hyperconvex.*

Therefore for a map $\tilde{\rho} : \tilde{V} \to T_\mu$ feasible to (3.14) and each $x \in V$, let $r(x) = \max_{e \in E_x} (d_\infty \circ \tilde{\rho})(e)/2$. Then, by the hyperconvexity (or the Helly property of balls), the intersection $\bigcap_{u \in V_x} B(\tilde{\rho}(u), r(x))$ of balls is nonempty. Take a point $p \in \bigcap_{u \in V_x} B(\tilde{\rho}(u), r(x))$, and let $\rho(x) := p$. Then $\tilde{\rho}(V_x) \subseteq B(\rho(x), r(x))$ for each $x \in V$. Now we arrive at a tight-span duality relation for the node-capacitated maximum multiflow problem, extending [12, Theorem 1.4]:

Theorem 3.8. *For a network (V,E,S,b,c), the maximum value of* (1.1) *is equal to the minimum value of the problem:*

$$
\begin{aligned}
(3.15) \qquad \text{Min.} \quad & \sum_{x \in V} 2b(x)r(x) + \sum_{xy \in E} c(xy) d_\infty(B(\rho(x), r(x)), B(\rho(y), r(y))) \\
\text{s.t.} \quad & \rho : V \to T_\mu, \ r : V \to \mathbf{R}_+, \\
& B(\rho(s), r(s)) \cap T_{\mu,s} \neq \emptyset \ (s \in S).
\end{aligned}
$$

Since $d_\infty(T_{\mu,s}, T_{\mu,t}) = \mu(s,t)$ [10], as in (3.2) one can see that every feasible (ρ, r) gives an upper bound of (1.1). Furthermore it is known that the case $\dim T_\mu \leq 1$ is exactly the case where μ is a tree distance.

Theorem 3.9 ([6] for metrics and [11] for general)**.** *For $\mu : \binom{S}{2} \to \mathbf{R}_+$, the following conditions are equivalent:*

(1) *μ is a tree distance.*

(2) $\dim T_\mu \leq 1$.

(3) *(T_μ, d_∞) is a metric-tree.*

(4) *For every* 4*-element subset* $\{s, t, s', t'\}$ *in* S*, we have*

$$\mu(s,t) + \mu(s',t') \leq \max \left\{ \begin{array}{c} \mu(s,t') + \mu(t,s'),\ \mu(s,s') + \mu(t,t'),\ \mu(s,t),\ \mu(s',t'), \\ \dfrac{\mu(t,s') + \mu(s',t') + \mu(t',t)}{2},\ \dfrac{\mu(s,s') + \mu(s',t') + \mu(t',s)}{2}, \\ \dfrac{\mu(s,t) + \mu(t,t') + \mu(t',s)}{2},\ \dfrac{\mu(s,t) + \mu(t,s') + \mu(s',s)}{2} \end{array} \right\}.$$

In this case, we can take $(T_\mu, \{T_{\mu,s}\}_{s \in S})$ as a continuous tree-realization of μ; in fact, it is a minimal realization of μ. So we get the problem (3.1). Note the 4-point characterization (4) will play an important role in Section 5.

4 Half-integral minimum cost multiflows

In this section, we prove a minimum cost version of the half-integrality assertion in Theorem 1.1. Let (V, E, S, b, c) denote a network with terminal weight μ. Further we are given a nonnegative *node-cost* $a : V \to \mathbf{Q}_+$. For a multiflow $f = (\mathcal{P}, \lambda)$, the *cost* $\mathrm{cost}(a, f)$ is defined by $\sum_{x \in V} a(x) \sum_{P \in \mathcal{P} : x \in VP} \lambda(P)$. We consider the following minimum cost multiflow problem:

(4.1) Maximize $\mathrm{val}(\mu, f) - \mathrm{cost}(a, f)$ over all multiflows f in (V, E, S, b, c).

The goal of this section is to prove the following mincost half-integrality:

Theorem 4.1. *Let* μ *be a tree distance on* S*. For any network* (V, E, S, b, c) *and any node-cost* $a : V \to \mathbf{Q}_+$*, there exists a half-integral optimal solution in* (4.1)*.*

In particular, letting $\mu \leftarrow p\mu$ for a sufficiently large $p > 0$, one can see that there is a half-integral maximum flow of the minimum total cost. In fact, $p = 2a(V)\{b(V) + E(V)\} + 1$ is sufficient as a consequence of this half-integrality; see the proof of Corollary 4.6 and the arguments in [3, 23].

As in Section 3.1, consider the extended graph $(\tilde{V}, \tilde{E})$. The mincost problem (4.1) is equivalent to (3.3) with adding cost $-\sum_{x \in V} a(x)\zeta(E_x)$ to the objective. The naive LP-dual is obtained from (3.4) by replacing $u(x) \geq l(e)$ with $u(x) \geq l(e) - a(x)$. Consequently, we get the metric-dual:

$$\text{(4.2)} \qquad \begin{array}{ll} \text{Min.} & \displaystyle\sum_{y \in V} b(x) \max_{e \in E_y} \{0, d(e) - a(y)\} + \sum_{e \in E} c(e) d(e) \\ \text{s.t.} & d : \text{metric on } \tilde{V}, \\ & d(s,t) \geq \mu(s,t) \quad (s, t \in S). \end{array}$$

Again this is a monotone metric-minimization.

Suppose that μ is realized by $(\Gamma, \{R_s\}_{s \in S}; 1)$ (by scaling). Let $\bar{\Gamma}$ denote the metric-tree corresponding to Γ, and let $\bar{R}_s$ denote the subtree in $\bar{\Gamma}$ corresponding to R_s. By Lemma 3.2 together with the argument after it, we get a geometric dual to (4.1):

$$\text{(4.3)} \qquad \begin{array}{ll} \text{Min.} & \displaystyle\sum_{y \in V} b(y) \max\{0, \mathrm{diam} F(y) - a(y)\} + \sum_{xy \in E} c(xy) d_{\bar{\Gamma}}(F(x), F(y)) \\ \text{s.t.} & F : V \to \mathcal{F}\bar{\Gamma}, \\ & F(s) \cap \bar{R}_s \neq \emptyset \quad (s \in S). \end{array}$$

Without loss of generality, we may assume that the network is node-only-capacitated ($c \to +\infty$) since the edge-capacity can be converted into node-capacity by subdividing each edge e and by defining the node-capacity of the subdivided node by $c(e)$. Further we assume that

(4.4) each terminal s is incident to only one inner node, and has a sufficiently large node-capacity $b(s) \to +\infty$ and zero cost $a(s) = 0$.

We can always make the input network satisfy this assumption. Indeed, for each terminal s, add new inner node s^*, join s^* and s, replace each edge xs incident to s by xs^*, set $(b(s^*), a(s^*)) := (b(s), a(s))$, and reset $(b(s), a(s)) := (+\infty, 0)$.

By a small perturbation, we may prove Theorem 4.1 under the assumption:

(4.5) $a(y)$ is positive for all inner nodes y.

Again we can take a small perturbation so that the perturbed problem solves the original problem and the bit size of the perturbation is not so large; see the proof of Corollary 4.6. Under the assumption, (4.3) becomes the following:

$$\begin{aligned} \text{(4.6)} \qquad \text{Min.} \quad & \sum_{y \in V \setminus S} b(y) \max\{0, \mathrm{diam} F(y) - a(y)\} \\ \text{s.t.} \quad & F : V \to \mathcal{F}\bar{\Gamma}, \\ & F(x) \cap F(y) \neq \emptyset \quad (xy \in E), \\ & F(s) \text{ is a single point in } \bar{R}_s \quad (s \in S). \end{aligned}$$

Our analysis is based on this problem. We need some notation. For a multiflow $f = (\mathcal{P}, \lambda)$ (in the original network, not in the extended network), the *flow-support* $\zeta^f : E \to \mathbf{R}_+$ is defined by $\zeta^f(e) := \sum\{\lambda(P) \mid P \in \mathcal{P}, e \in EP\}$ for $e \in E$. Recall the notation δy (the set of edges incident to node y). Then the following obvious relations are useful for us:

$$\begin{aligned} \zeta^f(\delta y) &= 2 \sum\{\lambda(P) \mid P \in \mathcal{P}, y \in VP\} \quad (y \in V \setminus S), \\ \sum_{y \in V \setminus S} \mathrm{diam} F(y) \zeta^f(\delta y)/2 &= \sum_{P \in \mathcal{P}} \lambda(P)(\mathrm{diam} F)(VP), \end{aligned}$$

where $(\mathrm{diam} F)(VP) = \sum_{x \in VP} \mathrm{diam} F(x)$ and we use $\mathrm{diam} F(s) = 0$ $(s \in S)$ in the second equation.

We derive an optimality criterion of primal-dual type. For a multiflow $f = (\mathcal{P}, \lambda)$ and a subtree map F feasible to (4.6), the duality gap is given by

$$\begin{aligned} & \sum_{y \in V \setminus S} b(y) \max\{0, \mathrm{diam} F(y) - a(y)\} - (\mathrm{val}(\mu, f) - \mathrm{cost}(a, f)) \\ = & \sum_{y \in V \setminus S} b(y) \max\{0, \mathrm{diam} F(y) - a(y)\} - \zeta^f(\delta y)(\mathrm{diam} F(y) - a(y))/2 \\ & \quad + \sum_{P \in \mathcal{P}} \lambda(P)\{(\mathrm{diam} F)(VP) - d_{\bar{\Gamma}}(\bar{R}_{s_P}, \bar{R}_{t_P})\}. \end{aligned}$$

Here we note $(\mathrm{diam} F)(VP) \geq d_{\bar{\Gamma}}(\bar{R}_{s_P}, \bar{R}_{t_P})$. Indeed, let $P = (x_0, x_1, \ldots, x_m)$ be an S-path. We can take points $p_i \in F^*(x_i) \cap F^*(x_{i+1})$ for $i = 0, 1, \ldots, m-1$ since $F^*(x_i) \cap F^*(x_{i+1})$ is nonempty. Then we have

$$\text{(4.7)} \qquad (\mathrm{diam} F)(VP) = \sum_{i=1}^{m-1} \mathrm{diam} F(x_i) \geq \sum_{i=1}^{m-1} d_{\bar{\Gamma}}(p_{i-1}, p_i) \geq d_{\bar{\Gamma}}(\bar{R}_s, \bar{R}_t).$$

An S-path P is said be *F-shortest* if $(\mathrm{diam} F)(VP) = d_{\bar{\Gamma}}(\bar{R}_{s_P}, \bar{R}_{t_P})(= \mu(s_P, t_P))$. Thus we have:

Lemma 4.2. *A multiflow f and a feasible subtree map F are both optimal if and only if*

(1) $\zeta^f(\delta y) = \begin{cases} 2b(y) & \text{if } \mathrm{diam} F(y) > a(y) \\ 0 & \text{if } \mathrm{diam} F(y) < a(y) \end{cases}$ *for $y \in V \setminus S$, and*

(2) *each path in f with positive flow-value is F-shortest.*

Let F^* be an optimal subtree map in (4.6). Replacing Γ by a larger (redundant) tree and by (3.8), we may assume (for simplicity) that

(4.8) Each subtree $F^*(x)$ is a ball with center $p_{F^*(x)}$ and radius $\mathrm{diam} F^*(x)/2$ so that $d_{\bar{\Gamma}}(p_{F^*(x)}, p) = \mathrm{diam} F^*(x)/2$ holds for every leaf p of $F^*(x)$.

In the sequel we construct an edge-weight $\zeta : E \to \mathbf{R}_+$ that can be represented as $\zeta = \zeta^f$ for some multiflow f satisfying (1-2). By Lemma 4.2 (1) and this positive cost assumption, there is no flow passing any inner node y with $\mathrm{diam} F^*(y) < a(y)$. Therefore we can delete all such inner nodes. In particular, $\mathrm{diam} F^*(y)$ is positive for all inner nodes y by (4.5). The shape of $F^*(y)$ gives us the important information of how flow passes through y. The condition (1) in Lemma 4.2 is a b-matching condition for the flow-support ζ^f. We study the condition (2). By (4.7) and (4.8), we get a local criterion for an S-path to be F^*-shortest:

Lemma 4.3. *An S-path $(s = x_0, x_1, x_2, \ldots, x_{m-1}, x_m = t)$ is F^*-shortest if and only if*

(1) *$F^*(x_i) \cap F^*(x_{i+1})$ is a leaf of both $F^*(x_i)$ and $F^*(x_{i+1})$ for $i = 0, 1, 2 \ldots, m-1$,*

(2) *$\bar{R}_s \cap F^*(x_1) = F^*(s)$, $\bar{R}_t \cap F^*(x_{m-1}) = F^*(t)$, and*

(3) *$F^*(x_{i-1})$ and $F^*(x_{i+1})$ belong to distinct connected components of $\bar{\Gamma} \setminus p_{F^*(x_i)}$ for $i = 1, 2, \ldots, m-1$.*

See Figure 3. In particular, we can delete all edges xy such that $F^*(x) \cap F^*(y)$ is not a leaf of both of $F^*(x)$ and $F^*(y)$; such an edge never has nonzero flow-support. Also we can delete a terminal s and its unique incident edge sx with $\bar{R}_s \cap F^*(x) \neq F^*(s)$. Motivated by (3), for an inner node y, let $\mathcal{X}^{F^*}(y)$ be the partition of δy defined as: xy and zy belong to the same part in $\mathcal{X}^{F^*}(y)$ if and only if $F^*(x)$ and $F^*(z)$ belong to the same connected component of $\bar{\Gamma} \setminus p_{F^*(y)}$ (well-defined by $\mathrm{diam} F^*(y) > 0$). Now consider the polyhedron $\mathcal{Q}$ formed by $\zeta : E \to \mathbf{R}_+$ satisfying the following inequalities:

(4.9) $\zeta(\delta y) = 2b(y) \quad (y \in V \setminus S : \mathrm{diam} F^*(y) > a(y)),$

(4.10) $0 \leq \zeta(\delta y) \leq 2b(y) \quad (y \in V \setminus S : \mathrm{diam} F^*(y) = a(y)),$

(4.11) $\zeta(U) \leq \zeta(\delta y \setminus U) \quad (y \in V \setminus S,\ U \in \mathcal{X}^{F^*}(y)).$

For every optimal multiflow f, its flow-support ζ^f satisfies these inequalities; in particular $\mathcal{Q}$ is nonempty. Indeed the first and the second correspond to Lemma 4.2 (1) and the capacity constraint, respectively. The third means that a flow coming from U must escape into $\delta y \setminus U$ by Lemma 4.3 (3). We say that inequality $\zeta(U) \leq \zeta(\delta y \setminus U)$ is *active* if the equality holds. Then the following property (for each $y \in V \setminus S$) is important:

(4.12) (1) If $|\mathcal{X}^{F^*}(y)| \geq 3$ and ζ is positive, then at most one inequality is active.

(2) If $|\mathcal{X}^{F^*}(y)| = 2$, then both two inequalities are always active.

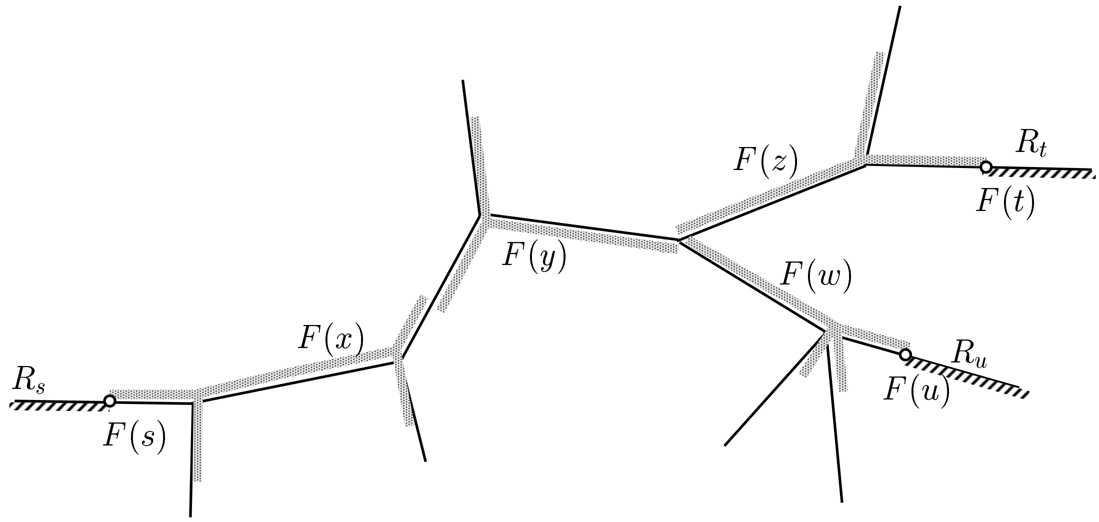

Figure 3: The image of F-shortest paths

This immediately follows from $\zeta(U) \leq \zeta(\delta y \setminus U) = \sum_{U' \in \mathcal{X}^{F^*}(y) \setminus \{U\}} \zeta(U')$. We establish the half-integrality of polyhedron $\mathcal{Q}$:

Lemma 4.4. *Every extreme point ζ^* of $\mathcal{Q}$ is half-integral, and moreover $\zeta^*(\delta x)$ is integral for $x \in V \setminus S$.*

Proof. Let ζ^* be an extreme point of $\mathcal{Q}$. We adapt the idea of Pap [34] expressing ζ^* as the projection of an extreme point of a fractional b-matching polyhedron for another graph G' with $\underline{b}, \overline{b}$.

Delete all edges e with $\zeta^*(e) = 0$. For each (inner) node y without active constraint in (4.11), set $(\underline{b}(y), \overline{b}(y)) = (0, 2b(y))$. Take an inner node y having active constraint $\zeta^*(U) = \zeta^*(\delta y \setminus U)$ for some $U \in \mathcal{X}^{F^*}(y)$. By (4.12) if $|\mathcal{X}^{F^*}(y)| \geq 3$, then such an active constraint is unique. If $|\mathcal{X}^{F^*}(y)| = 2$, then it also gives one equality constraint $\zeta^*(U) = \zeta^*(U^c)$ for $U^c := \delta y \setminus U \in \mathcal{X}^{F^*}(y)$. In particular, $\zeta^*(U) = \zeta^*(U^c) = \zeta^*(\delta y)/2$. Then split y into two nodes y_U and y_{U^c}. Reconnect each edge in U to y_U, and reconnect each edge in U^c to y_{U^c}. Let $(\underline{b}(y_U), \overline{b}(y_U)) = (\underline{b}(y_{U^c}), \overline{b}(y_{U^c})) = (0, b(y))$. Apply it to each inner node having an active constraint. Let $G' = (V', E')$ be the resulting graph with $\underline{b}, \overline{b}$. Then $\zeta^* : E \to \mathbf{R}_+$ can be extended to a fractional b-matching $\zeta^*\ E' \to \mathbf{R}_+$ by setting $\zeta^*(y_U y_{U^c}) = b(y) - \zeta^*(\delta y)/2$ for each inner node y having an active constraint.

The extended ζ^* is an extreme fractional b-matching in $G', \underline{b}, \overline{b}$. Indeed, suppose that $\zeta^* = (\zeta' + \zeta'')/2$ for some fractional b-matchings ζ', ζ''; we show $\zeta^* = \zeta' = \zeta''$. For a small $\epsilon > 0$ the restrictions of $(1-\epsilon)\zeta^* + \epsilon\zeta'$ and $(1-\epsilon)\zeta^* + \epsilon\zeta''$ to E are both points of $\mathcal{Q}$. Since the original ζ^* is extreme in $\mathcal{Q}$, for $e \in E$ we have $\zeta^*(e) = (1-\epsilon)\zeta^*(e) + \epsilon\zeta'(e) = (1-\epsilon)\zeta^*(e) + \epsilon\zeta''(e)$, and hence $\zeta^*(e) = \zeta'(e) = \zeta''(e)$. By $b(y) = \zeta^*(\delta y_U) = \zeta'(\delta y_U) = \zeta''(\delta y_U)$, we also have $\zeta^*(e) = \zeta'(e) = \zeta''(e)$ for $e \in E' \setminus E$.

Thus the original ζ^* is the projection of an extreme point the fractional b-matching polyhedron defined by integral node-capacity, and is half-integral by Lemma 2.1. Also $\zeta^*(\delta y)$ is integral (by $2\zeta^*(y_U y_{U^c}) = \zeta^*(\delta y)$). □

The next lemma completes the proof of Theorem 4.1.

Lemma 4.5. *For every extreme point ζ^* of $\mathcal{Q}$, there exists a half-integral multiflow f^* such that $\zeta^{f^*} = \zeta^*$ and each path in f^* is F^*-shortest. Moreover such a multiflow can be found in strongly polynomial time.*

Proof. We can construct such a multiflow in a greedy way. Delete all edges e with $\zeta^*(e) = 0$. Take a terminal s and its unique incident edge sx_1 with $\zeta^*(sx_1) > 0$. We extend sx_1 to an S-path as follows. By (4.11), there is a node x_2 incident to x_1 such that $\zeta^*(x_1x_2) > 0$, and sx_1 and x_1x_2 belong to distinct parts of $\mathcal{X}^{F^*}(x)$. Here, if some constraint $\zeta^*(U) \leq \zeta^*(\delta x_1 \setminus U)$ in (4.11) is active with $sx_1 \in \delta x_1 \setminus U$, then take x_1x_2 from U. Extend path (s, x_1) to (s, x_1, x_2). Consider node x_2. Suppose that x_2 is not a terminal. Again we can take a node x_3 incident to x_2 such that $\zeta^*(x_2x_3) > 0$, and x_1x_2 and x_2x_3 belong to distinct parts of $\mathcal{X}^{F^*}(x_2)$, as above. Extend path (s, x_1, x_2) to (s, x_1, x_2, x_3), and consider node x_3. Repeat it. By construction, this path induces a sequence of subtrees (balls) with properties (1-3) in Lemma 4.3, and thus meets each node at most once. After $O(|V|)$ steps, we arrive at some terminal $t = x_m$ and obtain an F^*-shortest S-path $P_1 = (s = x_0, x_1, x_2, x_3, \ldots, x_m = t)$. Let $\lambda(P_1)$ be defined as

$$\lambda(P_1) := \max\{\lambda \geq 0 \mid \zeta^*(e) - \lambda \text{ is nonnegative on } e \in EP_1 \text{ and satisfies (4.11)}\}.$$

Then we have

$$\begin{aligned} \lambda(P_1) &= \min\{\min_{e \in EP} \zeta^*(e), \min_{1 \leq i \leq m-1} \sigma_i\}, \\ \sigma_i &:= \min\left\{ \frac{\zeta^*(U) - \zeta^*(\delta x_i \setminus U)}{2} \;\middle|\; U \in \mathcal{X}^{F^*}(x_i), x_{i-1}x_i \notin U, x_ix_{i+1} \notin U \right\}. \end{aligned}$$

$\lambda(P_1)$ is positive half-integral since ζ^* is half-integral, and $\zeta^*(U) - \zeta^*(\delta x_i \setminus U)$ is positive integral by $\zeta^*(\delta x_i) \in \mathbf{Z}$ (Lemma 4.4) and by the choice of x_{i+1} at x_i together with (4.12).

Decrease ζ^* on EP_1 by $\lambda(P_1)$. Again ζ^* satisfies (4.9), (4.10), (4.11), is half-integral, and $\zeta^*(\delta y)$ is integral for each inner node y. Repeat this process until there is no edge in G. Then we get a half-integral optimal multiflow $f^* = (P_1, P_2, \ldots, P_k; \lambda(P_1), \lambda(P_2), \ldots, \lambda(P_k))$. These construction can be done in strongly polynomial time. Indeed, after getting path P_i, the number of edges decreases or some constraint in (4.11) for $U \in \mathcal{X}^{F^*}(y)$ becomes active. Once active, it keeps active in the sequent process (as long as U is nonempty). □

Corollary 4.6. *There exists a strongly polynomial time algorithm to find a half-integral optimal solution in* (4.1).

Proof. First, we need to take a perturbation $\epsilon : V \to \mathbf{R}_+$ so that $a + \epsilon$ is positive on $V \setminus S$, any optimal solution in (4.1) with cost $a + \epsilon$ is also optimal to (4.1) with cost a, and the bit size of ϵ is polynomially bounded. Suppose that μ is integer-valued. For cost a and flow f, we denote $\mathrm{val}(\mu, f) - \mathrm{cost}(a, f)$ by $\nu(a, f)$. Let Z be the set of inner nodes x with $a(x) = 0$. Take ϵ as $\epsilon(x) := 1/(2b(Z) + 1)$ for $x \in Z$ and zero otherwise; the bit size of ϵ is polynomially bounded. We show that ϵ is a desired one.

Let f^* be an optimal half-integral solution for (4.1) with cost a, and let f an optimal half-integral solution for (4.1) with cost $a + \epsilon$. Then $D := \nu(a, f^*) - \nu(a, f) (\geq 0)$ is half-integral, and we have

$$\begin{aligned} D &= \nu(a, f^*) - \nu(a + \epsilon, f^*) + \nu(a + \epsilon, f^*) - \nu(a + \epsilon, f) + \nu(a + \epsilon, f) - \nu(a, f) \\ &\leq \sum_{x \in Z} \epsilon(x)(\zeta^f - \zeta^{f^*}) \leq b(Z)/(2b(Z) + 1) < 1/2. \end{aligned}$$

Thus $D = 0$; f is also optimal to (4.1) with cost a. This perturbation technique is essentially due to Karzanov [23].

Solving (4.2) by Tardos' method [41], we obtain an optimal metric d^*. As in the proof of Lemma 3.2, we obtain an optimal subtree map F^*. Next construct $\mathcal{Q}$ from F^* as above; (4.9), (4.10), and (4.11) have a polynomial number of inequalities with

$\{-1,0,1\}$ coefficients. Take an extreme point ζ^* of $\mathcal{Q}$ again by Tardos' method. Finally, according to Lemma 4.5, we obtain a half-integral optimal multiflow f^* with $\zeta^{f^*} = \zeta^*$. These construction can be done in strongly polynomial time. □

Remark 4.7. Suppose that μ and a are integer-valued. By the integrality of μ, we can realize μ by a tree Γ with edge-length $\gamma = 1/2$ (Remark 2.2). In fact, we can take an optimal subtree map F^* such that each leaf of each $F^*(y)$ belongs to $V^1\Gamma$. Indeed, the rounding argument in Section 3.3 works. One can easily see that if $\mathrm{diam} F(x)$ is even then $\mathrm{diam} F_i(x) = \mathrm{diam} F(x)$ for each i, and if $k < \mathrm{diam} F(x) < k+2$ for even integer k then $k \leq \mathrm{diam} F_i(x) \leq k+2$ for each i. Therefore $\max\{0, \mathrm{diam} F(x)/2 - a(x)\} = \frac{1}{k}\sum_{i=1}^{k} \max\{0, \mathrm{diam} F_i(x)/2 - a(x)\}$ holds. Thus the maximum value of (4.1) is equal to the minimum value of the following *nonlinear* discrete subtree location problem:

$$
\begin{aligned}
(4.13) \qquad \text{Min.} \quad & \sum_{y \in V \setminus S} b(y) \max\{0, \mathrm{diam} F(y)/2 - a(y)\} \\
\text{s.t.} \quad & F : V \to \mathcal{F}\Gamma, \\
& F(x) \cap F(y) \neq \emptyset \quad (xy \in E), \\
& F(s) \text{ is a single point in } R_s \quad (s \in S).
\end{aligned}
$$

See Section 6 for related arguments.

Suppose that Γ is a path. Then $\mathcal{X}^{F^*}(y)$ is a bipartition and (4.11) is always active. From this, one can easily see that in the proof of Lemma 4.4 the graph G' is bipartite. Recall that the fractional b-matching polytope in a bipartite graph coincides with the (integral) b-matching polytope. Consequently any extreme solution ζ^* is integral, and moreover $\zeta^*(\delta y)$ is even for each inner node y. Thus we obtain an integral optimal multiflow. This explains the max-flow min-cut theorem in Example 1 in the introduction.

Corollary 4.8. *Suppose that μ is a tree distance realized by a path. Then there exists an integral optimal multiflow in* (4.1)*.*

However this is not a proper multiflow theorem. In fact there is a reduction to a minimum cost circulation problem. We omit it here; see [16, Section 3.4] for an essence.

5 Unbounded fractionality

The goal of this section is to prove that the set of tree distances is the only class admitting half-integrality theorem:

Theorem 5.1. *Suppose that $\mu : \binom{S}{2} \to \mathbf{Q}_+$ is not a tree distance. Then there is no positive integer k such that* (1.1) *has a $1/k$-integral optimal multiflow for every network* (V, E, S, b, c)*.*

Let k be an arbitrary positive integer. We construct a network in which its unique optimal multiflow is $1/k$-integral. Our construction is inspired by [22]; also see [24, section 3]. Let G_k be the graph of nodes $x_{i,j}$ $(1 \leq i,j \leq k)$ and edges $x_{i,j}x_{i+1,j}$ and $x_{i,j}x_{i,j+1}$ $(1 \leq i,j \leq k-1)$. Namely G_k is the $k \times k$ grid graph. Add nodes $s, t, s', t', \bar{s}, \bar{t}, \bar{s}', \bar{t}'$ and edges $s\bar{s}, t\bar{t}, s'\bar{s}', t'\bar{t}'$ to G_k, $\bar{s}x_{1,1}, \bar{s}x_{2,1}, \ldots, \bar{s}x_{k,1}$, $\bar{t}x_{1,k}, \bar{t}x_{2,k}, \ldots, \bar{t}x_{k,k}$, $\bar{s}'x_{1,1}, \bar{s}'x_{1,2}, \ldots, \bar{s}'x_{1,k}$, $\bar{t}'x_{k,1}, \bar{t}'x_{k,2}, \ldots, \bar{t}'x_{k,k}$. Let (V, E) be the resulting graph. Node-capacity b is defined as $b(x_{i,j}) = 1$, $b(\bar{s}) = b(\bar{t}) = 1$, $b(\bar{s}') = b(\bar{t}') = k-1$, and $b(s) = b(t) = b(s') = b(t') = +\infty$. Edge-capacity c is sufficiently large. A terminal set is defined by $\{s, t, s', t'\}$. Let

$(V, E, \{s, t, s', t'\}, b, c)$ be the resulting network. Consider the following (s, t)-paths P_i and (s', t')-paths Q_j:

$$\begin{aligned} P_i &= (s, \bar{s}, x_{i,1}, x_{i,2}, \ldots, x_{i,k}, \bar{t}, t) \quad (1 \leq i \leq k), \\ Q_j &= (s', \bar{s}', x_{1,j}, x_{2,j}, \ldots, x_{k,j}, \bar{t}', t') \quad (1 \leq j \leq k). \end{aligned}$$

Let $\mathcal{P} = \{P_1, P_2, \ldots, P_k, Q_1, Q_2, \ldots, Q_k\}$. The flow-value function λ is defined as $\lambda(P_i) = 1/k$ and $\lambda(Q_j) = (k-1)/k$ for each i, j. Then one can easily see:

(5.1) $f = (\mathcal{P}, \lambda)$ is a unique multiflow with the properties that

(i) it consists of (s, t)-paths and (s', t')-paths, and

(ii) nodes $\bar{s}, \bar{t}, \bar{s}', \bar{t}'$ are saturated.

In particular, there is no positive integer k that every 2-commodity flow maximization has a $1/k$-integral optimal multiflow.

Consider an arbitrary weight μ that is not a tree distance. Here we use the 4-point characterization (4) of tree distances in Theorem 3.9. There exists a 4-element set $\{s, t, s', t'\}$ satisfying

(5.2)
$$\mu(s,t) + \mu(s',t') > \max \left\{ \begin{array}{c} \mu(s,t') + \mu(t,s'),\ \mu(s,s') + \mu(t,t'),\ \mu(s,t),\ \mu(s',t'), \\ \dfrac{\mu(t,s') + \mu(s',t') + \mu(t',t)}{2},\ \dfrac{\mu(s,s') + \mu(s',t') + \mu(t',s)}{2}, \\ \dfrac{\mu(s,t) + \mu(t,t') + \mu(t',s)}{2},\ \dfrac{\mu(s,t) + \mu(t,s') + \mu(s',s)}{2} \end{array} \right\}.$$

Adding isolated terminals $S \setminus \{s, t, s', t'\}$ to $(V, E, \{s, t, s', t'\}, b, c)$, consider μ-weighted maximum multiflow problem (1.1) on (V, E, S, b, c). We show that the above-constructed multiflow $f = (\mathcal{P}, \lambda)$ is a unique optimum in this problem.

Contract all nodes in grid G_k into one node g, delete loops, and identify multiple edges. Set $b(g) = +\infty$. Then we obtain a new network of 9 nodes $g, s, t, s', t', \bar{s}, \bar{t}, \bar{s}', \bar{t}'$ and of 6 edges $s\bar{s}, t\bar{t}, s'\bar{s}', t'\bar{t}', g\bar{s}, g\bar{t}, g\bar{s}', g\bar{t}'$. There are only six S-paths in the new network. So any multiflow f can be represented as a 6-tuple $(f_{st}, f_{s't'}, f_{ss'}, f_{st'}, f_{ts'}, f_{tt'})$, where f_{uv} is the flow-value of a unique (u, v)-path for $u, v \in \{s, t, s', t'\}$. According to this contraction, the above-constructed multiflow f is transformed into a multiflow $f = (f_{st}, f_{s't'}, f_{ss'}, f_{st'}, f_{ts'}, f_{tt'}) = (1, k-1, 0, 0, 0, 0)$ in the new network. By (5.1), it suffices to show that f is a unique optimum in the new network.

Take an arbitrary optimal multiflow f' in the new network. Then $\min(f'_{st'}, f'_{ts'}) > 0$ is impossible. Otherwise, for small $\epsilon > 0$, decrease $f'_{st'}$ and $f'_{ts'}$ by ϵ and increase f'_{st} and $f'_{s't'}$ by ϵ. By $\mu(s,t) + \mu(s',t') > \mu(s,t') + \mu(t,s')$, the flow-value strictly increases. This contradicts the optimality. Similarly $\min(f'_{ss'}, f'_{tt'}) = 0$. Also $\min(f'_{ss'}, f'_{st'}) > 0$ is impossible. Otherwise, for small $\epsilon > 0$, decrease $f'_{st'}$ and $f'_{ss'}$ by ϵ and increase $f'_{s't'}$ by ϵ, and increase f'_{st} by 2ϵ; $\bar{t}$ is not saturated by $f'_{ts'} = f'_{tt'} = 0$. By $\mu(s,t) + \mu(s',t') > (\mu(s,t') + \mu(t',s') + \mu(s,s'))/2$, the flow-value increases. A contradiction. Consequently, at most one of $f'_{ss'}, f'_{st'}, f'_{ts'}, f'_{tt'}$ is positive. Suppose, say, $f'_{ss'} > 0$. Then both $\bar{t}$ and $\bar{t}'$ are not saturated. For small $\epsilon > 0$, decrease $f'_{ss'}$ by ϵ, and increase f'_{st} and $f'_{s't'}$ by ϵ. By $\mu(s,t) + \mu(s',t') > \mu(s,s') + \mu(t,t') \geq \mu(s,s')$, the flow-value increases. A contradiction. Moreover both $\mu(s,t)$ and $\mu(s',t')$ are positive by $\mu(s,t) + \mu(s',t') > \max(\mu(s,t), \mu(s',t'))$. So $\bar{s}, \bar{t}, \bar{s}', \bar{t}'$ are saturated by (s,t)-flows and (s',t')-flows. Consequently $f' = f$.

Remark 5.2. One can also prove that if μ is not a tree distance, then there is no positive integer k such that the minimum cost μ-weighted maximum multiflow problem has a $1/k$-integral optimal solution for every edge-only-capacitated network and every edge-cost. Indeed, take 4-element set $\{s, t, s', t'\} \subseteq S$ with property (5.2). Consider the network depicted in [22, Figure 3] (or [24, Figure 4]) with replacing $(1, 1', 2, 2')$ by (s, t, s', t'). By the same idea as above, one can show that the solution given in [22, p. 79] is a unique $1/k$-integral optimal solution.

6 Concluding remarks

In concluding, we discuss some related topics including future research directions.

Weighted generalizations of Mader's disjoint S-paths theorem. One can naturally consider the integer version of (1.1):

(6.1) Maximize $\mathrm{val}(\mu, f)$ over all *integer* multiflows f in (V, E, S, b, c),

where an *integer multiflow* is a multiflow for which its flow-value function is integer-valued. This generalizes Mader's disjoint S-paths packing problem [28, 29]. Hirai and Pap [17] considered (6.1) for the edge-only-capacitated case ($b \to +\infty$), and proved that if μ has a tree-realization $(\Gamma, \{R_s\}_{s \in S}; \gamma)$ so that *each subtree R_s is a ball*, then (6.1) is polynomial-time solvable and has a combinatorial min-max formula, which is also some kind of a tree location problem on Γ. Such a weight, a special tree distance, is called a *truncated tree metric*. Actually they proved a min-max formula and the polynomial-time solvability of the integer version of the mincost problem (4.1) for a tree metric μ; one can easily see that the problem (6.1) for a truncated tree metric reduces to it. This extends Mader's edge-disjoint S-path theorem [28] and its minimum cost generalization by Karzanov [21, 24], with solving one of his conjectures. They also proved that this result is tight, i.e., if μ is not a truncated tree metric, then (6.1) is NP-hard. Moreover Pap [35] proved that the node-capacitated (6.1) for a truncated tree metric is also polynomial-time solvable and admits a min-max formula. These developments were triggered by the tree-location method presented in this paper.

Convex-cost multiflows. Our method is applicable to the following class of *convex-cost* multiflow problems. Let (V, E) be an undirected graph with terminal set S. Instead of node- and edge-capacities, we are given monotone nondecreasing convex functions $g_x : \mathbf{R}_+ \to \mathbf{R} \cup \{+\infty\}$ $(x \in V)$ and $h_e : \mathbf{R}_+ \to \mathbf{R} \cup \{+\infty\}$ $(e \in E)$. We assume $g_x(0) < +\infty$ $(x \in V)$ and $h_e(0) < +\infty$ $(e \in E)$. A multiflow $f = (\mathcal{P}, \lambda)$ is a pair of a set $\mathcal{P}$ of S-paths and a flow-value function $\lambda : \mathcal{P} \to \mathbf{R}_+$ satisfying

$$g_x(\zeta^f(x)) < +\infty \quad (x \in V), \quad h_e(\zeta^f(e)) < +\infty \quad (e \in E),$$

where $\zeta^f : V \cup E \to \mathbf{R}_+$ is the flow-support defined by $\zeta^f(u) = \sum\{\lambda(P) \mid P \in \mathcal{P}, P \text{ contains } u\}$ for $u \in V \cup E$. Let μ be a terminal weight. Consider the following problem:

(6.2) $$\text{Maximize } \mathrm{val}(\mu, f) - \sum_{x \in V} g_x(\zeta^f(x)) - \sum_{e \in E} h_e(\zeta^f(e)) \text{ over all multiflows } f.$$

Recall the Fenchel duality theory in convex cost network flows [18, 37]. For a function $\varphi : \mathbf{R}_+ \to \mathbf{R} \cup \{+\infty\}$, the *conjugate function* $\varphi^* : \mathbf{R}_+ \to \mathbf{R} \cup \{+\infty\}$ is defined by

$$\varphi^*(p) := \sup_{\zeta \in \mathbf{R}_+} \{p\zeta - \varphi(\zeta)\} \quad (p \in \mathbf{R}_+).$$

If φ is nondecreasing convex, then so does φ^*. Suppose that μ is a tree distance realized by $(\Gamma, \{R_s\}_{s \in S}; 1)$. Let $\bar{\Gamma}$ be the metric-tree corresponding to Γ. By the Fenchel duality together with the argument in Sections 3.1 and 3.2, we obtain the following dual problem:

$$\text{(6.3)} \qquad \begin{array}{ll} \text{Min.} & \displaystyle\sum_{x \in V} g_x^*(\text{diam} F(x)) + \sum_{xy \in E} h_{xy}^*(d_{\Gamma}(F(x), F(y))) \\ \text{s.t.} & F : V \to \mathcal{F}\bar{\Gamma}, \\ & F(s) \cap \bar{R}_s \neq \emptyset \quad (s \in S). \end{array}$$

Under a mild condition on convex functions g_x, h_e, e.g., they are closed proper convex, the maximum value of (6.2) is equal to the minimum value of (6.3). Moreover, if all g_x, h_e are piecewise linear and their non-differentiable points are all integral, then there exists a half-integral integral multiflow in (6.2). To prove it, replace Lemma 4.2 (1) by a general kilta condition $g_x(\zeta^f(x)) + g_x^*(\text{diam} F(x)) = \zeta^f(x)\text{diam} F(x)$, and replace (4.9) and (4.10) by b-matching conditions corresponding to a vertical segment and a horizontal segment in the characteristic curve $\{(\zeta, h) \mid g_x(\zeta) + g_x^*(h) = \zeta h\}$. The remaining argument is the same as in Section 4, and an algorithm can also be obtained by the same way. Moreover, if each gradient of g_x, h_e is integer-valued, then (6.3) is discretized into a nonlinear discrete subtree location problem, as in Remark 4.7. The tight-span duality as in Section 3.4 is also straightforward. A detailed verification is left to readers.

Toward combinatorial polynomial algorithms. Our algorithm for problems (1.1), (1.2), (4.1) relies on a generic LP solver (ellipsoid method or interior point method). So it is challenge to find a *purely combinatorial* polynomial time algorithm. Goldberg and Karzanov [8] developed a combinatorial weakly polynomial time algorithm for the edge-only-capacitated minimum cost maximum free multiflow problem. Also Babenko and Karzanov [2] presented a combinatorial weakly polynomial time algorithm for the node-capacitated maximum free multiflow problem. One possible approach is to extend their algorithms. Our subtree location model might be some help to design an algorithm of a simpler dual description, such as: *Find an augmenting path in an auxiliary graph. If fails, then move, expand, or shrink subtrees until there appear new edges in the auxiliary graph.* Nevertheless, a half-integral flow-augmentation is quite nontrivial.

What is discrete convexity theory for multiflows ? Finally, let us mention a possibility of discrete convex analysis for multiflows. *Discrete convex analysis* [31, 32] is a theory of convex functions defined on integer points, aiming at a unified framework for well-solvable discrete optimizations, including network flows, matroids, and submodular functions. One of the motivations comes from combinatorial properties of convex functions arising from convex cost network (edge-only-capacitated) flows.

Since the single commodity weight μ is realized by a path, the dual (6.3) is a location problem on a path, and therefore it is the minimization of a function defined on the product of paths. The product of paths can be identified with a subset of integer lattice points $\mathbf{Z}^V$. Hence (6.3) can also be regarded as a minimization of a function defined on $\mathbf{Z}^V$. The objective function $g(p) := \sum_{xy \in E} h_{xy}^*(|p(x) - p(y)|)$ fulfills a certain submodular-type condition and is a typical example of *L-convex functions*, a fundamental class of *discrete convex functions* in discrete convex analysis.

It would be interesting to extend the theory of discrete convexity to include multicommodity flows and well-solvable discrete location problems. In general tree distance weights, the dual is the minimization of the function $g(p) := \sum_{xy} h_{xy}^*(d_{\Gamma}(p(x), p(y)))$ defined on the *product of trees*; the similarity to the L-convex function above is notable.

Moreover, in the node-capacitated case, it is a minimization of a function defined on the product of folder complexes; see Remark 3.4. Probably, discrete convex analysis for multiflows would be a theory of convex functions defined on the product of trees, more generally, graph structures, or polyhedral complexes, beyond integer lattice points in the Euclidean space. Such a theory, if exists, might bring a novel paradigm to discrete optimization. We will pursue this issue in the subsequent papers.

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