The maximum multiflow problems with bounded fractionality

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Multicommodity flows

\((G, c)\): undirected network
\(G = (VG, EG), c : EG \rightarrow \mathbb{Z}_+\)
\(H\): commodity graph \((VH \subseteq VG)\)

**Definition**

\[ Multiflow \ f = \{\varphi_{st} \mid st \in EH\}, \text{ where } \varphi_{st}: (\text{fractional}) \ st\text{-flow}, \]
\[
\sum_{st \in EH} |\varphi_{st}(e)| \leq c(e) \quad (e \in EG).\]

*Communication networks, transportation, VLSI, LP-relaxations of NP-hard problems*
*Edge-disjoint paths, Multicut, 0-extension, Sparsest cut, ...*
The maximum multiflow problem (G, c): network, H: commodity graph

Total flow-value of multiflow f

\[ \| f \| := \sum \{ f_{st} \mid st \in EH \} \]

**Maximum Multiflow Problem**

Maximize \( \| f \| \) over all multiflows \( f \) in \((G, c; H)\)

**Fractionality**

\( \text{frac}(H) := \) the least positive integer \( k \) with property:

\[ \exists 1/k\text{-integral maximum flow for } (\forall G, \forall c; H). \]

\( \text{frac}(\ | ) = 1 \) \hspace{1cm} \text{(Ford-Fulkerson 56)}

\( \text{frac}(\ || ) = 2 \) \hspace{1cm} \text{(Hu 63)}

\( \text{frac}(\ ||| \cdots | ) = \frac{K_n}{k \geq 3} \)

\( \text{frac}(\ | | ) = 2 \) \hspace{1cm} \text{(Karzanov 98)}

\( \text{frac}(\ | | | ) = \frac{K_2 + K_n}{4}, \frac{K_n}{2} \) \hspace{1cm} \text{(Lomonsov 04)}

\( \text{frac}(\ | | | | ) = \frac{K_n}{2} \) \hspace{1cm} \text{(Lomonsov 04)}
Maximum Multiflow Problem

\((G, c): \text{ network, } H: \text{ commodity graph}\)

Total flow-value of multiflow \(f\)

\(\|f\| := \sum \{f_{st} \mid st \in EH\}\)

Maximum Multiflow Problem

Maximize \(\|f\|\) over all multiflows \(f\) in \((G, c; H)\)

Fractionality

\(\text{frac}(H) := \text{ the least positive integer } k \text{ with property: }\)
\(\exists 1/k\text{-integral maximum flow for } (\forall G, \forall c; H).\)

\(\text{frac}(\mid) = 1\) \hspace{2cm} \text{(Ford-Fulkeson 56)}
\(\text{frac}(\mid\mid) = 2\) \hspace{2cm} \text{(Hu 63)}
\(\text{frac}(\mid\mid\mid \cdots \mid) = +\infty\) \hspace{2cm} \text{for } k \geq 3
\(\text{frac}(\triangle) = \text{frac}(\boxtimes) = \text{frac}(K_n) = 2\) \hspace{2cm} \text{(Lovasz 76, Cherkassky 77)}
\(\text{frac}(\|\triangle\|) = 2\) \hspace{2cm} \text{(Karzanov 98)}
\(\text{frac}(\|\boxtimes\|) = \text{frac}(K_2 + K_n) = 4\) \hspace{2cm} \text{(Lomonsov 04)}
\(\text{frac}(\triangle \triangle) = ?\)

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**Karzanov’s conjecture**

**Fractionality problem (Kazarnov, ICM Kyoto, 90)**

*Classify commodity graphs having *finite* fractionality.*

(Property P) For every intersecting maximal stable sets $A, B, C$ of $H$

$$A \cap B = B \cap C = C \cap A.$$ 

**Theorem (Karzanov 89)**

If $\frac{1}{\text{frac}(H)} < +\infty \Rightarrow H$ satisfies P.

**Conjecture (Karzanov, ICM Kyoto, 90)**

If $H$ satisfies P \(\Rightarrow\)

1. $\frac{1}{\text{frac}(H)} < +\infty$,
2. $\text{frac}(H) \in \{1, 2, 4\}$. 
A weighted generalization

\((G, c)\): network, \(S \subseteq VG\): terminal set, \(\mu: (\binom{S}{2}) \rightarrow \mathbb{R}_+:\) terminal weight

Multiflow \(f = \{\varphi_{st} \mid st \in \binom{S}{2}\}\)

Flow-value \(\|f\|_\mu := \sum\{\mu(st)f_{st} \mid st \in \binom{S}{2}\}\).

\(\mu\)-weighted maximum multiflow problem

Maximize \(\|f\|_\mu\) over all multiflows \(f\) in \((G, c; S)\)

- 0-1 weight \(\Leftrightarrow\) commodity graph \(H\)
- \(\text{frac}(\mu):=\) the least positive integer \(k\) s.t. \(\cdots\)
A weighted generalization

\((G, c):\text{network}, \ S \subseteq V_G: \text{terminal set}, \ \mu : \binom{S}{2} \to \mathbb{R}_+: \text{terminal weight}\)

Multiflow \(f = \{\phi_{st} \mid st \in \binom{S}{2}\}\)

Flow-value \(\|f\|_\mu := \sum \{\mu(st)f_{st} \mid st \in \binom{S}{2}\}\).

\(\mu\)-weighted maximum multiflow problem

\textbf{Maximize} \(\|f\|_\mu\) \text{ over all multiflows } f \text{ in } (G, c; S)

- 0-1 weight \(\iff\) commodity graph \(H\)
- \(\frac{\mu}{\mu} := \text{the least positive integer } k \text{ s.t. } \cdots\)

\textbf{Theorem (Karzanov 98 for metrics, H. 09 for general)}

If \(\frac{\mu}{\mu} < +\infty \Rightarrow \dim T_\mu \leq 2\),

\textbf{Tight span (Isbell 64, Dress 84)}

\(T_\mu := \text{Minimal } \{p \in \mathbb{R}_+^{S} \mid p(s) + p(t) \geq \mu(s, t) \ s, t \in S\}\)

\textbf{Remark (H. 09)}

\(H\) satisfies property \(P \iff \dim T_{\mu_H} \leq 2\)
Main Theorem

Main Theorem (H. 09, STOC 2010)

If \( \dim T_\mu \leq 2 \),

1. \( \exists \) 1/12-integral maximum multiflow
   in \( \mu \)-max multiflow problem for every Eulerian network,
   \((\rightarrow \exists 1/24\text{-integral max multiflow for every network})\)

2. \( \exists \) strongly polytime algorithm to find it \textit{provided} \( \mu \) is 0-1.

Corollary

- \( \frac{\mu}{2} \in \{1, 2, 3, 4, 6, 8, 12, 24, +\infty\} \)
- \( \frac{\mu}{2} < +\infty \Leftrightarrow \dim T_\mu \leq 2 \)
- \( \frac{H}{2} < +\infty \Leftrightarrow \text{Property P} \)

\textit{We do not know whether 1/24 is tight} !!
Proof Idea

Synchronizing Primal & Dual

**Dual**: LP-dual $\rightarrow$ minimum $O$-extension on a *folder complex*

**Primal**: *Fractional* splitting-off & potential update
Proof Idea

Synchronizing Primal & Dual

**Dual**: LP-dual → *minimum O-extension on a folder complex*

**Primal**: Fractional splitting-off & potential update

A key concept: folder complex (≈ 2-dimensional tight span)

\[ \mathcal{K} \text{: metrized polygonal complex obtained by gluing } folders: \]

\[ (\mathbb{R}^2, l_1) \sim \]

\[ \begin{array}{c}
\text{(Chepoi 00)} \\
\mathcal{K} \text{: folder complex} \overset{\text{def}}{=} \left\{ \begin{array}{c}
\mathcal{K} \text{ is simply-connected, and} \\
\text{link graph of each vertex has girth } \geq 8. \\
\iff CAT(0) \text{ under } l_2\text{-metrization.}
\end{array} \right\}
\end{array} \]
Folder complex and multiflow combinatorial duality

\[ \mu = \begin{array}{cccc}
  s & t & u & v \\
  3 & 3 & 3 & 2 \\
  t & 4 & 4 & 1 \\
  u & 2 & 3 & \\
  v & 1 & \\
\end{array} \]

Theorem (H. 09)

Suppose \( \exists \) folder complex \( \mathcal{K} \) with normal regions \( \{ R_s \}_{s \in S} \) s.t.
\[
\mu(s, t) = d_{\mathcal{K}, l_1}(R_s, R_t) \quad (s, t \in S).
\]

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Folder complex and multiflow combinatorial duality

\[ \rho : VG \rightarrow VK \]

\[ \mathbf{max}_{f} \| f \|_{\mu} = \min \sum_{xy \in EG} c(xy)d_{K,l}(\rho(x), \rho(y)) \]

\begin{align*}
\text{s.t. } & \rho : VG \rightarrow VK, \rho(s) \in R_s \quad \leftarrow \text{potential} \\
\end{align*}

- \( \dim T_{\mu} \leq 2 \) if and only if \( \mu \) is realized by a folder complex.
- Optimal potential \( \rho \) is obtained by solving LP-dual.

**Theorem (H. 09)**

Suppose \( \exists \) folder complex \( K \) with normal regions \( \{R_s\}_{s \in S} \) s.t.

\[ \mu(s, t) = d_{K,l}(R_s, R_t) \quad (s, t \in S). \]

Suppose \( \exists \) folder complex \( K \) with normal regions \( \{R_s\}_{s \in S} \) s.t.

\[ \mu(s, t) = d_{K,l}(R_s, R_t) \quad (s, t \in S). \]
Splitting-off

- fork: $\tau = (e, y, e')$

Equation:
$$e, y, e' \quad \Rightarrow \quad \text{Diagram}$$
Splitting-off

- fork: \( \tau = (e, y, e') \)

\[ e \quad y \quad e' \]

2-subdivision

Proposition (H. 09)

\((G; c)\): Eulerian, \(V_G\) is optimal potential, \(y\): inner node

If \((y) =\), then \(y\) has a splittable fork.

Approach (after some preprocessing on terminals)

\((G; c; C)\)!! \((G_0; c_0; C_0)\) until \((V_G) =\) with \((G_0; c_0; C_0)\) Eulerian

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Splitting-off

- fork: $\tau = (e, y, e')$

Proposition (H. 09)

$(G, c)$: Eulerian, $\rho : VG \rightarrow VK$: optimal potential, $y$: inner node
If $\rho(y) = \bullet$, then $y$ has a splittable fork.

Approach (after some preprocessing on terminals)

$(G, c)$: Eulerian, $\rho$: optimal potential
$(G, c; \rho) \rightarrow \cdots \rightarrow (G', c'; \rho')$ until $\rho(V^{\text{in}}G') = \bullet$ with $(G', kc')$ Eulerian
Fractional splitting-off & potential update

- **Fractional splitting-off**

\[
\tau = (e, x, e')
\]

\[
G
\]

\[
x
\]

\[
ee
\]

\[
G^\tau
\]

\[
x
\]

\[
e
\]

\[
e'
\]

\[
c(e^\tau) = 2 - \alpha
\]

- **Maximum possible value:**

\[
(c \cdot d^\rho := \sum_{xy \in E_G} c(xy) d_K(\rho(x), \rho(y)))
\]

\[
\alpha(\tau) = \min \left\{ \frac{c \cdot d^{\rho'}}{d^{\rho'}(e^\tau)} \mid \rho' : \text{potential}, d^{\rho'}(e^\tau) > 0 \right\}
\]

Update \((G, c; \rho) \rightarrow (G^\tau, c; \rho'), c(e^\tau) := 2 - \alpha(\tau)\)
Fractional splitting-off & potential update

- Fractional splitting-off & potential update

• Maximum possible value: 

\[
\alpha(\tau) = \min \left\{ \frac{c \cdot d^\rho' - c \cdot d^\rho}{d^\rho'(e^\tau)} \middle| \begin{array}{l}
\rho' : \text{potential}, \\
\quad d^\rho'(e^\tau) > 0
\end{array} \right\} \in 2\mathbb{Z}_+ \quad \text{if Eulerian at } \rho^{-1}(\bullet, \bullet)
\]

Update \((G, c; \rho) \to (G^\tau, c; \rho'), c(e^\tau) := 2 - \alpha(\tau)\)

We can take \(\rho'\) so that potentials move toward \(\bullet\)
Fractional splitting-off & potential update

- Fractional splitting-off & potential update

Maximum possible value:

\[ (c \cdot d^\rho := \sum_{xy \in EG} c(xy)d_{K}(\rho(x), \rho(y))) \]

\[ \alpha(\tau) = \min \left\{ \frac{c \cdot d^{\rho'} - c \cdot d^\rho}{d^{\rho'}(e^\tau)} \left| \begin{array}{l}
\rho' : \text{potential, } d^{\rho'}(e^\tau) > 0 \\
\end{array} \right. \right\} \in 2\mathbb{Z}^+ \text{ if Eulerian at } \rho^{-1}(\bullet, \bullet) \text{, } 1, 2, 3, \text{ or } 4 \]

Update \((G, c; \rho) \rightarrow (G^\tau, c; \rho'), c(e^\tau) := 2 - \alpha(\tau)\)

We can take \(\rho'\) so that potentials move toward \(\bullet\).

\(\rightarrow\) Capacities of edges between \(\rho^{-1}(\bullet)\) cancel out.

\(\rightarrow\) We can make \((G, c; \rho)\) so that \(\rho(V^{\text{in}}G) = \bullet\) and \((G, 12c)\) is Eulerian.
Concluding remarks

- we do not know whether 24 is tight
  ... conjectured tight upper bound is 4.
- proof is lengthy and complicated
- polytime algorithm provided the size of $\mathcal{K}$ is fixed
  ($\mu$ is 0-1 $\Rightarrow$ $\exists O(|S|^2)$-size $\mathcal{K}$ realizing $\mu$).
- many new classes of $\frac{\text{frac}}{2} = 2, 4$
- CAT(0)-complex
- application to approximation algorithms for integer multiflows?
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Thank you for your attention!