A bipartite graph $G = (V^+, V^-; A)$ with $V^+ \cup V^- \neq \emptyset$ is said to be *DM-irreducible* if it cannot be decomposed into more than one nonempty component in the DM-decomposition. Otherwise, it is called *DM-reducible.* A graph G with $V^+ = \emptyset$ or $V^- = \emptyset$ is DM-irreducible, as the whole graph is a (vertical or horizontal) tail. Note that G with $A = \emptyset$, $V^+ \neq \emptyset$ and $V^- \neq \emptyset$ is DM-reducible, as it can be decomposed into two nonempty components, the horizontal tail $G_0 = (V^+, \emptyset; \emptyset)$ and the vertical tail $G_\infty = (\emptyset, V^-; \emptyset)$.

The following theorem is a reformulation of the result due to Marcus–Minc [186] and Brualdi [22] (see also Brualdi–Ryser [24, Theorem 4.2.2]).

Theorem 2.2.24. *For a bipartite graph $G = (V^+, V^-; A)$ with $|V^+| = |V^-|$ the following three conditions are equivalent:*
(i) *G is DM-irreducible,*
(ii) $C(G) = \{(V^+, \emptyset), (\emptyset, V^-)\}$,
(iii) $\nu(G \setminus \{u, v\}) = \nu(G) - 1$ *for* $\forall u \in V^+$, $\forall v \in V^-$.

Proof. The equivalence between (i) and (ii) follows from Theorem 2.2.22(1).
For (ii) $\Rightarrow$ (iii), first note (ii) implies $\nu(G) = |V^+|$, and hence $\nu(G \setminus \{u, v\}) \leq |V^+| - 1 = \nu(G) - 1$. Take a minimum cover (W^+, W^-) of $G \setminus \{u, v\}$. Since $(W^+ \cup \{u\}, W^- \cup \{v\})$ is a cover of G but not a minimum cover, we see $\nu(G) + 1 \leq |W^+ \cup \{u\}| + |W^- \cup \{v\}| = \nu(G \setminus \{u, v\}) + 2$.
For (iii) $\Rightarrow$ (ii) suppose that (ii) fails. We divide into two cases: (a) $\nu(G) = |V^+|$ and (b) $\nu(G) < |V^+|$. In case (a), there exists $(U^+, U^-) \in C(G)$ such that $U^+ \neq \emptyset$ and $U^- \neq \emptyset$. For $u \in U^+$ and $v \in U^-$, $(U^+ \setminus \{u\}, U^- \setminus \{v\})$ is a cover of $G \setminus \{u, v\}$. Hence $\nu(G \setminus \{u, v\}) \leq |U^+ \setminus \{u\}| + |U^- \setminus \{v\}| = \nu(G) - 2$. In case (b), both horizontal and vertical tails are nonempty. For $u \in V_0^+$ and $v \in V_\infty^-$ we have $\nu(G \setminus \{u, v\}) = \nu(G)$ by Corollary 2.2.23(2). ■

The concept of DM-decomposition may be extended to matrices by means of the DM-decomposition of associated bipartite graphs. Recall from §2.2.1 that for a matrix $A = (A_{ij})$ the associated bipartite graph is defined by $G = (V^+, V^-; \tilde{A})$ with $V^+ = \mathrm{Col}(A)$, $V^- = \mathrm{Row}(A)$ and $\tilde{A} = \{(j, i) \mid A_{ij} \neq 0\}$. A DM-component $G_k = (V_k^+, V_k^-; A_k)$ of G corresponds to the submatrix $A[V_k^-, V_k^+]$, which will be referred to as a *DM-component* of A.

The following relation is obvious from the definitions, but provides the DM-decomposition with a linear algebraic significance.

Proposition 2.2.25. term-rank $A = \nu(G)$. □

The *DM-decomposition* of A gives the finest block-triangularization of a matrix by means of a transformation $P_r A P_c$ using two permutation matrices P_r and P_c, where it is imposed that each diagonal block in a block-triangularization has full term-rank. In fact, Theorem 2.2.22(1) combined with Proposition 2.2.25 above guarantees this term-rank condition for the diagonal blocks produced by the DM-decomposition. Furthermore, term-rank

coincides with rank for a generic matrix, in which all nonzero entries are independent parameters (cf. Proposition 2.1.12). Hence, for a generic matrix, the DM-decomposition gives the finest proper block-triangularization in the sense of §2.1.4.

For instance, the matrix version of the DM-decomposition of the graph in Fig. 2.7 is given by

$$
A = \begin{array}{c|ccc|c|cc|c|}
 & & V_0^+ & & V_1^+ & V_2^+ & & V_\infty^+ \\ \hline
V_0^- & t_1 & t_2 & t_3 & & & & \\
 & & t_4 & t_5 & t_6 & t_7 & & \\
V_1^- & & & & t_8 & & & \\
V_2^- & & & & & t_9 & t_{10} & \\
 & & & & & t_{11} & t_{12} & t_{13} \\
V_\infty^- & & & & & & & t_{14} \\
 & & & & & & & t_{15} \\ \hline
\end{array} .
$$

Note that term-rank $A[V_k^-, V_k^+] = \min(|V_k^+|, |V_k^-|)$ for $k = 0, 1, 2, \infty$. The matrix $P_r A P_c$ of (2.16) in Example 2.1.15 gives an instance of the DM-decomposition of a term-nonsingular matrix.

Though term-rank is a natural combinatorial counterpart, it is not the same as rank, which is undoubtedly more important in applications. A numerical (nongeneric) matrix may or may not have the same rank as term-rank, and accordingly, the DM-decomposition may or may not be a proper block-triangular form. The present argument shows the following.

Proposition 2.2.26. *For a matrix A the following three conditions* (i)–(iii) *are equivalent.*
(i) rank A = term-rank A.
(ii) *The DM-decomposition is a proper block-triangularization, i.e., the DM-components $A[V_k^-, V_k^+]$ $(k = 0, 1, \cdots, b, \infty)$ satisfy*

$$\mathrm{rank}\, A[V_k^-, V_k^+] = \min(|V_k^+|, |V_k^-|) \qquad (k = 0, 1, \cdots, b, \infty).$$

(iii) *There exist $I \subseteq \mathrm{Row}(A)$ and $J \subseteq \mathrm{Col}(A)$ such that* $\mathrm{rank}\, A[I, J] = 0$, $\mathrm{rank}\, A[\mathrm{Row}(A) \setminus I, J] = |\mathrm{Row}(A) \setminus I|$, *and* $\mathrm{rank}\, A[I, \mathrm{Col}(A) \setminus J] = |\mathrm{Col}(A) \setminus J|$.

Proof. The equivalence of (i) and (ii) is immediate from Theorem 2.2.22. For (iii) take $I = \mathrm{Row}(A) \setminus V_0^-$ and $J = \mathrm{Col}(A) \setminus V_0^+$. ■

The concept of DM-irreducibility can be naturally defined for matrices, and it coincides with the well-studied concept of full indecomposability (cf. Brualdi–Ryser [24] and Schneider [288]). To see this, first recall that a matrix A is said to be ***fully indecomposable*** if it does not contain a zero submatrix $A[I, J] = O$ with $I \neq \emptyset$, $J \neq \emptyset$ and $|I| + |J| = \max(|\mathrm{Row}(A)|, |\mathrm{Col}(A)|)$. Since $A[I, J] = O$ if and only if $(V^- \setminus I, V^+ \setminus J)$ is a cover of the associated graph $G = (V^+, V^-; \tilde{A})$, matrix A is fully indecomposable if and only if G has no cover (U^+, U^-) such that $U^+ \neq V^+$, $U^- \neq V^-$ and

$|U^+| + |U^-| = \min(|V^+|, |V^-|)$. The latter condition is equivalent to the DM-irreducibility (cf. Theorem 2.2.24). Henceforth we use DM-irreducibility as a synonym of full indecomposability.

The DM-irreducibility for square generic matrices admits two further characterizations in addition to those given in Theorem 2.2.24. The first says that the DM-irreducibility for a generic matrix is equivalent to the inverse matrix having a completely dense nonzero pattern.

Theorem 2.2.27. *A square generic matrix A is DM-irreducible if and only if A is nonsingular and $(A^{-1})_{ji} \neq 0$ for all (j, i).*

Proof. Since $(A^{-1})_{ji} = \det A[R \setminus \{i\}, C \setminus \{j\}] / \det A$, where $R = \mathrm{Row}(A)$ and $C = \mathrm{Col}(A)$, the claim here reduces to the equivalence of (i) and (iii) in Theorem 2.2.24. ■

The determinant of a generic matrix A can be regarded as a polynomial in the nonzero entries. Specifically, let $\mathcal{T}$ denote the set of nonzero entries of A, which is algebraically independent over a ground field $\mathbf{K}$. Then $\det A \in \mathbf{K}[\mathcal{T}]$, where $\mathbf{K}[\mathcal{T}]$ means the ring of polynomials in $\mathcal{T}$ over $\mathbf{K}$.

The following theorem gives an algebraic characterization of the DM-irreducibility in terms of the irreducibility of the determinant as a multivariate polynomial. This is proven in Ryser [285] and credited essentially to Frobenius [78] in Ryser [286].

Theorem 2.2.28. *A square generic matrix A is DM-irreducible if and only if $\det A$ is an irreducible (nonzero) polynomial in $\mathbf{K}[\mathcal{T}]$, where $\mathcal{T}$ denotes the set of nonzero entries of A.*

Proof. The "if" part is obvious, since, for a DM-reducible A with no tails, $\det A$ is equal to the product of the determinants of the diagonal blocks of the DM-decomposition of A (and $\det A = 0$ if a nonempty tail exists). For the "only if" part assume that $\det A$ is factored as $\det A = f_1 \cdot f_2$ with $f_1, f_2 \in \mathbf{K}[\mathcal{T}] \setminus \mathbf{K}$. For $k = 1, 2$, let $\mathcal{T}_k$ denote the set of the variables of $\mathcal{T}$ that appear in f_k. Put

$$R_k = \{i \in R \mid A_{ij} \in \mathcal{T}_k\}, \quad C_k = \{j \in C \mid A_{ij} \in \mathcal{T}_k\} \qquad (k = 1, 2),$$

where $R = \mathrm{Row}(A)$ and $C = \mathrm{Col}(A)$. Then $R_1 \cap R_2 = \emptyset$, $R_1 \cup R_2 = R$, $C_1 \cap C_2 = \emptyset$, $C_1 \cup C_2 = C$ for $k = 1, 2$, since for each pair of terms in f_1 and f_2 their product remains in $f_1 \cdot f_2 = \det A$ as a nonvanishing term, which in turn corresponds to a perfect matching in the associated bipartite graph. We may assume $|R_1| \geq |C_1| \geq 1$ without loss of generality. If $A[R_1, C_2] = O$, A is DM-reducible. If $A_{ij} \neq 0$ for some $i \in R_1$ and $j \in C_2$, the variable $t = A_{ij}$ cannot appear in $\det A$, since otherwise t must be contained in f_k for $k = 1$ or 2, which implies $i \in R_k$ and $j \in C_k$, a contradiction to $R_1 \cap R_2 = \emptyset$ and $C_1 \cap C_2 = \emptyset$. The disappearance of t in $\det A$ implies $\det A[R \setminus \{i\}, C \setminus \{j\}] = 0$, or equivalently, $\nu(G \setminus \{i, j\}) < \nu(G) - 1$ in terms of the associated bipartite graph G. This shows the DM-reducibility by Theorem 2.2.24. ■

Remark 2.2.29. We have derived the DM-decomposition in a systematic manner on the basis of the Jordan–Hölder-type theorem for submodular functions, though alternative quicker derivations would have been possible. Our systematic derivation here enables us to generalize the DM-decomposition to a more sophisticated decomposition in §4.4, called the CCF (combinatorial canonical form) of layered mixed matrices. The DM-decomposition serves as one of the main tools for the graph-theoretic methods for systems analysis (see Duff–Erisman–Reid [59], Murota [204, Chaps. 2 and 3]), whereas the CCF is for the matroid-theoretic methods to be developed in Chap. 4 and Chap. 6. Applications of the DM-decomposition can be found in Ashcraft–Liu [8], Erisman–Grimes–Lewis–Poole–Simon [73], Hellerman–Rarick [109, 110], O'Neil–Szyld [255], and Pothen–Fan [273]. □

20 Algorithms and Combinatorics

Matrices and Matroids for Systems Analysis

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