MATHEMATICAL RESEARCH

Volume 87

Structural Approach in Systems Analysis by Mixed Matrices — An Exposition for Index of DAE —

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ICIAM 95

ICIAM 95

Proceedings of the Third International Congress on Industrial and Applied Mathematics held in Hamburg, Germany, July 3–7, 1995

edited by
Klaus Kirchgässner
Oskar Mahrenholtz
Reinhard Mennicken

Akademie Verlag

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Abstract

We explain a matroid-theoretic combinatorial approach to the structural analysis of engineering systems by referring, as an example, to the problem of computing the index of a system of differential-algebraic equations (DAEs). The mathematical framework in this approach classifies the coefficients appearing in the equations into two categories, independent physical parameters and dimensionless fixed constants, and describes engineering systems by means of the mixed polynomial matrices. Furthermore, it resorts to a kind of dimensional analysis. It is emphasized that relevant physical observations are crucial to successful mathematical modelling for structural analysis.

0 Introduction

Modern engineering systems consist of a huge number of interconnected elementary components. Such large-scale systems cannot be expected to function properly as a whole unless they are well structured in some appropriate sense. The overall behaviors of such (lumped-constant) systems are governed by the way the components are connected as well as by the way individual components behave.

It is now recognized that the structural, or qualitative, aspects of large-scale engineering systems can be treated successfully by combinatorial mathematics such as graph-, network- and matroid-theory. Typically, the structural approach focuses on the qualitative properties of systems by disregarding the numerical values of system parameters through certain mathematical assumptions on the genericity of the numerical values involved.

In this paper we explain a matroid-theoretic method in the structural approach to engineering systems by taking for example the problem of computing the index of a system of differential-algebraic equations (DAEs) (see the following section for the definition of "index").

Extensive study has been made recently on this problem (Brenan–Campbell–Petzold [BCP], Elmqvist–Otter–Cellier [EOC], Gear [G1,G2], Hairer–Wanner [HW]). A number of "structural algorithms" based on graph-theoretic techniques have been developed (Bujakiewicz [B], Bujakiewicz–van den Bosch [BB], Duff–Gear [DG], Pantelides [P], Ungar–Kröner–Marquardt [UKM]), followed by a refinement of this approach called "combinatorial relaxation" (Iwata–Sakuta–Murota [IMS], Murota [M8]). It is accepted that structural considerations should be useful and effective in practice for this problem and that the generic values computed by "structural algorithms" will have practical significance. At the same time, however, the limitation of the structural approach has also been realized. Structural algorithms, ignoring numerical data, may fail to render the correct answer if numerical cancellations do occur for some reason or other. Already Pantelides [P] recognized this phenomenon and very recently Ungar–Kröner–Marquardt [UKM] expounded this point referring to an example problem that arises from an analysis of distillation columns in chemical engineering.

The matroid-theoretic approach (Iri [I], Murota [M3], Recski [R1]) to be described in this paper has been proposed as a partial remedy for such shortcoming of the existing graph-theoretic "structural algorithms." On the basis of the observation that the failure of the "structural algorithms" is often caused by the numerical cancellation among fixed constants, the matroid-theoretic approach classifies the numbers characterizing engineering systems (e.g., the coefficients appearing in the equations) into two categories, independent physical parameters and dimensionless fixed constants. Furthermore, it resorts to a kind of dimensional analysis. Mathematically, this amounts to relaxing the primitive genericity assumption in the graph-theoretic method and describing engineering systems by means of "mixed polynomial matrices" rather than by "structured polynomial matrices."

We intend this paper to be an expository article for engineers by using simple examples as the main tool for exposition and by keeping the formal mathematics to the minimum. Though we feature the index problem of DAE, the methodology expounded here is more general in scope. It is emphasized that relevant physical observations are crucial to successful mathematical modelling for structural analysis.

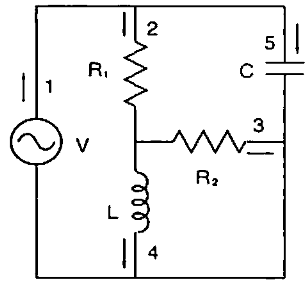


Figure 1: An electrical network

1 Structural Approach to Index of DAE

1.1 Index of differential-algebraic equations

Let us start with a simple electrical network[1] of Fig. 1 to introduce the concept of index of a system of differential-algebraic equations (DAEs) and to explain a graph-theoretic structural algorithm. The network consists of a voltage source V (branch 1), two ohmic resistors R_1 and R_2 (branch 2 and branch 3), an inductor L (branch 4), and a capacitor C (branch 5). This network can be described, in the frequency domain, by a system of equations $A^{(1)}x = b$, where $x = (\xi_1, \cdots, \xi_5, \eta_1, \cdots, \eta_5)^{\mathrm{T}}$ is a 10 dimensional vector representing currents ξ_i in and the voltage η_i across branch i $(i = 1, \cdots, 5)$, $b = (0,0,0,0,0; V, 0,0,0,0)^{\mathrm{T}}$ is another 10 dimensional vector representing the source, and $A^{(1)}$ is a 10×10 matrix given by

$$A^{(1)} = \begin{array}{|ccccc|ccccc|}
\multicolumn{1}{c}{\xi_1} & \xi_2 & \xi_3 & \xi_4 & \multicolumn{1}{c}{\xi_5} & \eta_1 & \eta_2 & \eta_3 & \eta_4 & \multicolumn{1}{c}{\eta_5} \\
\hline
1 & -1 & 0 & 0 & -1 & & & & & \\
-1 & 0 & 1 & 1 & 1 & & & & & \\
\hline
 & & & & & -1 & -1 & 0 & -1 & 0 \\
 & & & & & 0 & 1 & 1 & 0 & -1 \\
 & & & & & 0 & 0 & -1 & 1 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & R_1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & R_2 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & sL & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & sC \\
\hline
\end{array} . \qquad (1.1)$$

[1]This network was produced by Professor Rafael Huber from the Universitat Politecnica de Catalunya in Barcelona, Spain, with the observation that the usual graph-theoretic structural algorithm fails for it, as will be explained later in this paper. It was communicated to the author by Francois Cellier and Pawel Bujakiewicz.

As usual, s is the variable for the Laplace transformation. The first two equations, corresponding to the 1st and 2nd rows of $A^{(1)}$, represent Kirchhoff's current law (KCL), while the following three equations Kirchhoff's voltage law (KVL). The last five equations (rows of $A^{(1)}$) express the element characteristics (constitutive equations). The equation $A^{(1)}x = b$ is a so-called (linear time-invariant) DAE, a mixture of differential equations and algebraic equations, since the coefficient matrix $A^{(1)}$ contains the variable s that corresponds to $\mathrm{d}/\mathrm{d}t$ (differentiation with respect to time).

For a linear time-invariant DAE in general, say $Ax = b$ with $A = A(s)$ being a nonsingular polynomial matrix in s, the index is defined by[2]

$$\nu(A) = \max_{i,j} \deg\,(A^{-1})_{ji} + 1. \tag{1.2}$$

Here it should be clear that each entry $(A^{-1})_{ji}$ of A^{-1} is a rational function in s and the degree of a rational function p/q (with p and q polynomials) is defined by $\deg\,(p/q) = \deg p - \deg q$. An alternative expression is

$$\nu(A) = \max_{i,j} \deg\,((i,j)\text{-cofactor of } A) - \deg\det A + 1. \tag{1.3}$$

When $\deg A_{ij} = 0$ or 1 for all (i,j) with $A_{ij} \neq 0$, the index $\nu(A)$ agrees with the index of nilpotency of A as a matrix pencil. Note that $\nu(A)$ can be defined also for a rational function matrix A by the above formulas.

The solution x to $Ax = b$ is of course given by $x = A^{-1}b$, and therefore $\nu(A) - 1$ equals the highest order of the derivatives of the input b that can possibly appear in the solution x. As such, a high index indicates the difficulty in numerical solution of the DAE, and sometimes even the inadequacy in mathematical modelling. See Brenan–Campbell–Petzold [BCP], Gear [G1,G2], Hairer–Wanner [HW], Ungar–Kröner–Marquardt [UKM] for more about the index of DAE.

1.2 Structural approach

In the graph-theoretic structural approach we extract the information about the degree of the entries of the matrix, ignoring the numerical values of the

[2] The definition of index given here applies only to linear time-invariant DAE systems. Index can be defined for more general systems and two kinds are distinguished in the literature, differential index and perturbation index, which coincide with each other for linear time-invariant DAE systems.

coefficients. Associated with the matrix $A^{(1)}$ of (1.1) for the electrical network, for example, we consider

$$A^{(1)}_{\mathrm{str}} = \begin{array}{c} \begin{array}{ccccc|ccccc} \xi_1 & \xi_2 & \xi_3 & \xi_4 & \xi_5 & \eta_1 & \eta_2 & \eta_3 & \eta_4 & \eta_5 \end{array} \\ \left|\begin{array}{ccccc|ccccc} t_1 & t_2 & 0 & 0 & t_3 & & & & & \\ t_4 & 0 & t_5 & t_6 & t_7 & & & & & \\ \hline & & & & & t_8 & t_9 & 0 & t_{10} & 0 \\ & & & & & 0 & t_{11} & t_{12} & 0 & t_{13} \\ & & & & & 0 & 0 & t_{14} & t_{15} & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & t_{16} & 0 & 0 & 0 & 0 \\ 0 & t_{17} & 0 & 0 & 0 & 0 & t_{18} & 0 & 0 & 0 \\ 0 & 0 & t_{19} & 0 & 0 & 0 & 0 & t_{20} & 0 & 0 \\ 0 & 0 & 0 & s\,t_{21} & 0 & 0 & 0 & 0 & t_{22} & 0 \\ 0 & 0 & 0 & 0 & t_{23} & 0 & 0 & 0 & 0 & s\,t_{24} \end{array}\right| \end{array} \qquad (1.4)$$

where $t_1, \cdots, t_{24}$ are assumed to be independent parameters.

For a nonsingular polynomial (or rational) matrix $A = A(s)$ in general we consider a matrix A_{str}, called the structured matrix associated with A, in a similar manner. Namely, we define the (i, j) entry of A_{str} by

$$(A_{\mathrm{str}})_{ij} = \begin{cases} t_{ij} s^{\deg A_{ij}} & A_{ij} \neq 0 \\ 0 & A_{ij} = 0 \end{cases} \qquad (1.5)$$

where t_{ij} is an independent parameter. We refer to the index of A_{str} in the sense of (1.2) or (1.3) as the structural index of A and denote it by $\nu_{\mathrm{str}}(A)$, namely,

$$\nu_{\mathrm{str}}(A) = \nu(A_{\mathrm{str}}). \qquad (1.6)$$

Though there is no guarantee that the structural index $\nu_{\mathrm{str}}(A)$ agrees with the true index $\nu(A)$, the structural index has a computational advantage that it can be computed by an efficient combinatorial algorithm that is free from numerical difficulties. Specifically, we consider a bipartite graph $G(A) = (R, C; E)$ with the left vertex set corresponding to the row set R of A, the right vertex set corresponding to the column set C of A, and the edge set corresponding to the set of nonzero entries of A, i.e.,

$$E = \{(i, j) \mid i \in R, j \in C, A_{ij} \neq 0\}.$$

Each edge $(i, j) \in E$ is given a weight $w_{ij} = \deg \det A_{ij}$. Then the formula (1.3) shows that

$$\nu_{\mathrm{str}}(A) = \max_{|M_{n-1}|=n-1} w(M_{n-1}) - \max_{|M_n|=n} w(M_n) + 1, \qquad (1.7)$$

where $n = |R| = |C|$ and $\max_{|M_k|=k} w(M_k)$ denotes the maximum weight of a matching of size k (for $k = n-1, n$). (A matching M is a set of edges such that no two members of M have an end-vertex in common, and the weight of M, denoted $w(M)$, is defined by $w(M) = \sum_{(i,j)\in M} w_{ij}$.) Thus the structural index $\nu_{\rm str}$ can be computed by solving weighted bipartite matching problems; see Bujakiewicz [B] and Bujakiewicz–van den Bosch [BB] for an algorithmic enhancement of this idea.

For instance, the bipartite graph $G(A^{(1)})$ associated with our example matrix $A^{(1)}$ of (1.1) (with $n = 10$) is given in Fig. 2, where, in the first figure, the thin lines indicate edges of weight 0 and the thick lines designate two edges, $(i,j) = (9,4), (10,10)$ of weight 1, and in the second figure the thick lines indicate the edges of a maximum-weight matching of size 10. We have $\max w(M_{n-1}) = \max w(M_n) = 2$ and therefore $\nu_{\rm str}(A^{(1)}) = 2 - 2 + 1 = 1$, whereas $\nu(A^{(1)}) = 2 - 1 + 1 = 2$. The discrepancy between $\nu_{\rm str}(A^{(1)})$ and $\nu(A^{(1)})$ is ascribed to the numerical cancellation in the expansion of $\det A^{(1)}$, which is evidenced by the discrepancy between $\deg\det A^{(1)}_{\rm str} = 2$ and $\deg\det A^{(1)} = 1$.

A closer look at this phenomenon reveals that this cancellation is *not an accidental cancellation but a cancellation with good reason* which could be better called *structural cancellation.* In fact, we can identify the 3×3 singular submatrix in the middle of the coefficient matrix for the KVL,

η_2	η_3	η_4
−1	0	−1
1	1	0
0	−1	1

as the reason for this cancellation. It is emphasized that the physical parameters R_1, R_2, L, C are treated as mutually independent parameters, which cannot be cancelled out among themselves.

1.3 An embarrassing phenomenon

The failure of the structural approach for our example matrix is no surprise at all. But it is observed further by Pawel Bujakiewicz (private communication) that the structural index of our example problem varies depending on how KVL is described. This phenomenon makes us reconsider the meaning of the structural index, as is discussed subsequently.

In expressing KVL we now take the loop 1–5 (V–C) instead of the loop 1–2–4 (V–R_1–L), which corresponds to the 4th row of $A^{(1)}$, to obtain a second

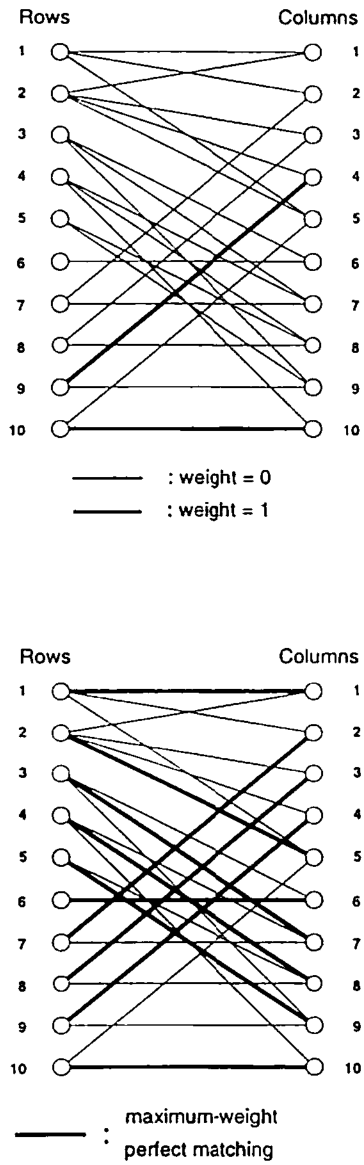


Figure 2: Graph $G(A^{(1)})$ and the maximum-weight matching

description of the same electrical network. The coefficient matrix of the second description is given by

$$
A^{(2)} = \begin{array}{c}
\begin{array}{ccccc|ccccc}
\xi_1 & \xi_2 & \xi_3 & \xi_4 & \xi_5 & \eta_1 & \eta_2 & \eta_3 & \eta_4 & \eta_5
\end{array} \\
\left[\begin{array}{ccccc|ccccc}
1 & -1 & 0 & 0 & -1 & & & & & \\
-1 & 0 & 1 & 1 & 1 & & & & & \\ \hline
 & & & & & -1 & 0 & 0 & 0 & -1 \\
 & & & & & 0 & 1 & 1 & 0 & -1 \\
 & & & & & 0 & 0 & -1 & 1 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & R_1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & R_2 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & sL & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & sC
\end{array}\right]
\end{array}, \qquad (1.8)
$$

which differs from $A^{(1)}$ in the 3rd row. The associated structured matrix $A^{(2)}_{\rm str}$, defined by (1.5), differs from $A^{(1)}_{\rm str}$ also in the 3rd row.

Naturally we have $\nu(A^{(1)}) = \nu(A^{(2)}) = 2$. It turns out, however, that $\nu_{\rm str}(A^{(1)}) = 1$ and $\nu_{\rm str}(A^{(2)}) = 2$, where the latter is computed as $\nu_{\rm str}(A^{(2)}) = \nu(A^{(2)}_{\rm str}) = 2 - 1 + 1 = 2$ according to the expression (1.7).

This example demonstrates that the structural index is not determined uniquely by a physical/engineering system, but it depends on its mathematical description. It is emphasized that

$$
A^{(1)} : \begin{array}{c}
\begin{array}{ccccc} \eta_1 & \eta_2 & \eta_3 & \eta_4 & \eta_5 \end{array} \\
\left[\begin{array}{ccccc}
-1 & -1 & 0 & -1 & 0 \\
0 & 1 & 1 & 0 & -1 \\
0 & 0 & -1 & 1 & 0
\end{array}\right]
\end{array}
\quad \text{and} \quad
A^{(2)} : \begin{array}{c}
\begin{array}{ccccc} \eta_1 & \eta_2 & \eta_3 & \eta_4 & \eta_5 \end{array} \\
\left[\begin{array}{ccccc}
-1 & 0 & 0 & 0 & -1 \\
0 & 1 & 1 & 0 & -1 \\
0 & 0 & -1 & 1 & 0
\end{array}\right]
\end{array}
\qquad (1.9)
$$

are equally a legitimate choice of the description of KVL and there is nothing inherent to distinguish between the two. In this way the structural index is vulnerable to our innocent choice.

2 What is Combinatorial Structure?

In view of the above example we have to question the physical relevance of the structural index, as defined in (1.6), and reconsider how we should recognize the combinatorial structure of physical systems. The proposed mathematical framework is based on two different physical observations; the one is the distinction between "accurate" and "inaccurate" numbers (Section 2.1), and the other is the consistency with respect to physical dimensions (Section 2.3).

2.1 Two kinds of numbers

Let us continue with our electrical network. The matrix $A^{(1)}$ of (1.1) is a matrix pencil written as

$$A^{(1)}(s) = A_0^{(1)} + sA_1^{(1)}. \tag{2.1}$$

We observe here that the nonzero entries of the coefficient matrices $A_k^{(1)}$ ($k = 0, 1$) are classified into two groups: one group of fixed constants (± 1) and the other group of system parameters R_1, R_2, L and C. Accordingly, we can split $A_k^{(1)}$ ($k = 0, 1$) into two parts:

$$A_k^{(1)} = Q_k^{(1)} + T_k^{(1)} \qquad (k = 0, 1) \tag{2.2}$$

with

$$Q_0^{(1)} = \left[\begin{array}{ccccc|ccccc} 1 & -1 & 0 & 0 & -1 & & & & & \\ -1 & 0 & 1 & 1 & 1 & & & & & \\ \hline & & & & & -1 & -1 & 0 & -1 & 0 \\ & & & & & 0 & 1 & 1 & 0 & -1 \\ & & & & & 0 & 0 & -1 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \quad T_0^{(1)} = \left[\begin{array}{ccccc|ccccc} 0 & 0 & 0 & 0 & 0 & & & & & \\ 0 & 0 & 0 & 0 & 0 & & & & & \\ \hline & & & & & 0 & 0 & 0 & 0 & 0 \\ & & & & & 0 & 0 & 0 & 0 & 0 \\ & & & & & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & R_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & R_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

$$Q_1^{(1)} = \left[\begin{array}{ccccc|ccccc} 0 & 0 & 0 & 0 & 0 & & & & & \\ 0 & 0 & 0 & 0 & 0 & & & & & \\ \hline & & & & & 0 & 0 & 0 & 0 & 0 \\ & & & & & 0 & 0 & 0 & 0 & 0 \\ & & & & & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \quad T_1^{(1)} = \left[\begin{array}{ccccc|ccccc} 0 & 0 & 0 & 0 & 0 & & & & & \\ 0 & 0 & 0 & 0 & 0 & & & & & \\ \hline & & & & & 0 & 0 & 0 & 0 & 0 \\ & & & & & 0 & 0 & 0 & 0 & 0 \\ & & & & & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & L & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & C \end{array}\right].$$

When concrete numbers are given to the physical parameters, R_1, R_2, L, C, those numbers are not expected to be exactly equal to their nominal values, but they lie in certain intervals of real numbers of engineering tolerance. Even when both R_1 and R_2 are specified to be 1Ω, for example, their actual values will be something like $R_1 = 1.02\Omega$ and $R_2 = 0.99\Omega$.

Generally, when a physical system is described by a polynomial matrix

$$A(s) = \sum_{k=0}^{N} s^k A_k, \tag{2.3}$$

it is often justified (see Section 2.2) to assume that the nonzero entries of the coefficient matrices A_k $(k = 0, 1, \cdots, N)$ are classified similarly into two groups. In other words, we can distinguish the following two kinds of numbers, together characterizing a physical system. We may refer to the numbers of the first kind as "fixed constants" and to those of the second kind as "system parameters."

Accurate Numbers (fixed constants): Numbers accounting for various sorts of conservation laws such as Kirchhoff's laws which, stemming from topological incidence relations, are precise in value (often ± 1), and therefore cause no serious numerical difficulty in arithmetic operations on them.

Inaccurate Numbers (system parameters): Numbers representing independent physical parameters such as resistances in electrical networks and masses in mechanical systems which, being contaminated with noise and other errors, take values independent of one another, and therefore can be modelled as algebraically independent numbers.

See Section 2.2 for another example and Murota–Iri [MI] and Murota [M3] (Chap. 4) for further discussions on this distinction of numbers.

The above observation leads to the assumption that the coefficient matrices A_k $(k = 0, 1, \cdots, N)$ in (2.3) are expressed as

$$A_k = Q_k + T_k \qquad (k = 0, 1, \cdots, N), \tag{2.4}$$

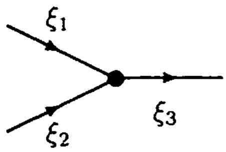
where

(A-Q1): Q_k $(k = 0, 1, \cdots, N)$ are matrices over $\mathbb{Q}$ (the field of rational numbers), and

(A-T): The collection $\mathcal{T}$ of nonzero entries of T_k $(k = 0, 1, \cdots, N)$ is algebraically independent over $\mathbb{Q}$.

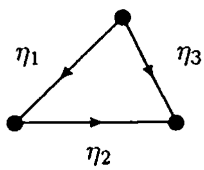
Namely, each A_k is assumed to be a *mixed matrix*, which terminology will be defined formally in Section 3. Then $A(s)$ is split accordingly into two parts:

$$A(s) = Q(s) + T(s) \tag{2.5}$$



$$-1 \cdot \xi_1 - 1 \cdot \xi_2 + 1 \cdot \xi_3 = 0$$

KCL



$$-1 \cdot \eta_1 - 1 \cdot \eta_2 + 1 \cdot \eta_3 = 0$$

KVL

$$2 \cdot H_2O = 2 \cdot H_2 + 1 \cdot O_2$$

Stoichiometry

Velocity-displacement $\quad v = 1 \cdot \dot{x} \ (= s \cdot x)$

Current-charge $\quad \xi = 1 \cdot \dot{Q} \ (= s \cdot Q)$

Figure 3: Accurate numbers

with

$$Q(s) = \sum_{k=0}^{N} s^k Q_k, \qquad T(s) = \sum_{k=0}^{N} s^k T_k. \tag{2.6}$$

Namely, $A(s)$ is a *mixed polynomial matrix* in the terminology of Section 3.

Our intention in this splitting (2.4) or (2.5) is to extract a more meaningful combinatorial structure from the matrix $A(s)$ by treating the Q-part numerically and the T-part symbolically. This is based on the following observations:

Q-part: The nonzero pattern of the Q-matrices is subject to our arbitrary choice in the mathematical description, as we have seen in our example network, and hence the structure of the Q-part should be treated numerically, or linear-algebraically. In fact this is feasible in practice, since the entries of the Q-matrices are usually small integers, causing no serious numerical difficulty in arithmetic operations.

T-part: The nonzero pattern of the T-matrices is relatively stable against our arbitrary choice in the mathematical description of the constitutive equations and therefore it can be regarded as representing some aspect of the combinatorial structure of the system. It can be treated properly by graph-theoretic concepts and algorithms.

Combination: The structural information from the Q-part and the T-part can be combined properly and efficiently by virtue of the fact that each part defines a well-behaved and well-studied combinatorial structure called matroid. Mathematical and algorithmic results from matroid theory afford effective methods of system analysis. See Section 3.

Remark 2.1. It should be clear that assuming algebraic independence of $\mathcal{T}$ is equivalent to regarding the members of $\mathcal{T}$ as independent parameters, and therefore to considering the family of systems parametrized by those parameters in $\mathcal{T}$. □

Remark 2.2. The rationality of the entries of Q_k is not essential. In case nonrational constants are involved, we may replace $\mathbb{Q}$ with an appropriate extension field K, though the field K affects the computational complexity of algorithms. See Section 3. □

2.2 Descriptor form rather than standard form

In the literature of control theory, DAE systems appear as descriptor equations. Here we will explain that the descriptor form is much more suitable for representing the combinatorial structure than the standard state-space form (which corresponds to ODE systems in the normal form).

We use another example, a simple mechanical system (see Fig. 4) which consists of two masses m_1, m_2, two springs k_1, k_2, and a damper f; u is the force exerted from outside.

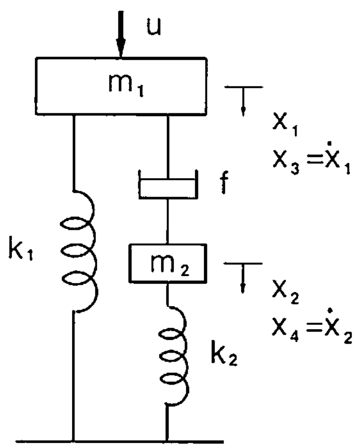


Figure 4: A mechanical system

We may describe the system in the form of state-space equations (Kalman [K])

$$\dot{x}(t) = \hat{A}x(t) + \hat{B}u(t) \tag{2.7}$$

in terms of $x = (x_1, \ldots, x_4)$, where x_1 (resp. x_2) is the displacement of mass m_1 (resp. m_2), x_3 (resp. x_4) is its velocity and

$$\hat{A} = \begin{array}{c} \\ \end{array}\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k_1/m_1 & 0 & -f/m_1 & f/m_1 \\ 0 & -k_2/m_2 & f/m_2 & -f/m_2 \end{array}, \quad \hat{B} = \begin{array}{c} u \\ \hline 0 \\ 0 \\ 1/m_1 \\ 0 \end{array}.$$

The state-space equations (2.7) have been useful for investigating analytic and algebraic properties of a dynamical system, and the structural or combinatorial analysis was started by Lin [L1] based on it. It is now recognized, however, that the state-space equations are not very suitable for representing the combinatorial structure of a system in that the entries of matrices $\hat{A}$ and $\hat{B}$ of (2.7) are usually not independent but interrelated to one another, being subject to algebraic relations.

In this respect, the so-called descriptor form (Luenberger [L2])

$$\bar{F}\dot{x}(t) = \bar{A}x(t) + \bar{B}u(t), \tag{2.8}$$

or its Laplace transform

$$s\bar{F}\hat{x}(s) = \bar{A}\hat{x}(s) + \bar{B}\hat{u}(s),$$

is more suitable. Then a system is described by a polynomial matrix

$$A(s) = (\bar{A} - s\bar{F} \mid \bar{B}). \tag{2.9}$$

For our mechanical system it may be natural to introduce two additional variables x_5 (= force by the damper f) and x_6 (= relative velocity of the two masses), where $x_5 = fx_6$ and $x_6 = \dot{x}_1 - \dot{x}_2$, to describe it in the descriptor form (2.8)[3]. We then have

$$A(s) = \begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & u \end{array} \\ \left[\begin{array}{ccccccc} -s & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -s & 0 & 1 & 0 & 0 & 0 \\ -k_1 & 0 & -sm_1 & 0 & -1 & 0 & 1 \\ 0 & -k_2 & 0 & -sm_2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & f & 0 \\ -s & s & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array} \tag{2.10}$$

as the matrix of (2.9). Note that no complicated algebraic expressions are involved in this matrix, for which it is reasonable to assume (A-Q1) and (A-T) above. Consequently, $A(s)$ of (2.10) is expressed as $A(s) = Q(s) + T(s)$ with

$$Q(s) = \begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & u \end{array} \\ \left[\begin{array}{ccccccc} -s & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -s & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ -s & s & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}, \quad T(s) = \begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & u \end{array} \\ \left[\begin{array}{ccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -k_1 & 0 & -sm_1 & 0 & 0 & 0 & 0 \\ 0 & -k_2 & 0 & -sm_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & f & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}.$$

Here we have $\mathcal{T} = \{m_1, m_2, k_1, k_2, f\}$ as the set of system parameters.

It is emphasized that the coefficient matrices $\hat{A}$ and $\hat{B}$ in the standard state-space form do not admit such natural splitting into two parts.

[3] We could replace the equation $x_6 = \dot{x}_1 - \dot{x}_2$ by $x_6 = x_3 - x_4$, which may be more natural. Our choice is to make the example less trivial.

2.3 Dimensional analysis

The second physical observation, due to Murota [M1], is a kind of dimensional analysis concerning the "accurate numbers," i.e., concerning the constant part $Q(s) = \sum_{k=0}^{N} s^k Q_k$ of the matrix $A(s)$ in (2.5).

The "accurate numbers" usually represent topological and/or geometrical incidence coefficients (cf. Fig. 3), which have no physical dimensions, so that it is natural to expect that the entries of Q_k are dimensionless constants. On the other hand, the variable (indeterminate) s should have the physical dimension of the inverse of time, since it corresponds to the differentiation with respect to time. This implies, in particular, that each entry of the term $s^k Q_k$ has the physical dimension of time with exponent $-k$.

Next we consider the physical dimensional consistency in the system of equations $A(s)x = b$, where $A(s)$ is assumed to be $m \times n$. Since this system is to represent a physical system, relevant physical dimensions are associated with both the variables (corresponding to the components of x) and the equations (corresponding to the components of b), or alternatively, with the columns and the rows of the matrix $A(s)$. Choosing time as one of the fundamental dimensions, we denote by $-c_j$ and $-r_i$ the exponent to the dimension of time associated respectively with the jth column and the ith row. The principle of dimensional homogeneity then demands that [Dimension of ith row] = [Dimension of (i,j) entry] $\times$ [Dimension of jth column] for each (i,j) with $A_{ij} \neq 0$. In particular, the (i,j) entry of $A(s)$, as well as that of $Q(s)$, should have the dimension of time with exponent $c_j - r_i$.

Combining these two facts on the dimension of the (i,j) entry, we obtain

$$r_i - c_j = k \qquad \text{if} \quad (Q_k)_{ij} \neq 0, \tag{2.11}$$

or in matrix form,

$$Q(s) = \mathrm{diag}\,[s^{r_1}, \ldots, s^{r_m}] \cdot Q(1) \cdot \mathrm{diag}\,[s^{-c_1}, \ldots, s^{-c_n}]. \tag{2.12}$$

It follows from this decomposition that every nonvanishing subdeterminant of $Q(s)$ is a monomial in s over $\mathbb{Q}$.

We have thus arrived at a class of polynomial matrices for representing the structure of linear time-invariant dynamical systems. Namely, we consider polynomial matrices $A(s)$ in indeterminate s with rational coefficients which are represented as

$$A(s) = Q(s) + T(s), \tag{2.13}$$

where

(A-Q2): Every nonvanishing subdeterminant of $Q(s)$ is a monomial in s over $\mathbb{Q}$, and

(A-T): The collection $\mathcal{T}$ of the nonzero coefficients of the entries of $T(s)$ is algebraically independent over $\mathbb{Q}$.

For our mechanical system we may choose time T, length L and mass M as the fundamental quantities in the dimensional analysis. Then the dimensions of velocity and force are given by $T^{-1}L$ and $T^{-2}LM$, respectively. The physical dimensions associated with the equations, i.e., with the rows of $A(s)$ of (2.10), are

velocity	velocity	force	force	force	velocity
$T^{-1}L$,	$T^{-1}L$,	$T^{-2}LM$,	$T^{-2}LM$,	$T^{-2}LM$,	$T^{-1}L$

whereas those with the variables (x_i and u), i.e., with the columns of $A(s)$, are

length	length	velocity	velocity	force	velocity	force
L,	L,	$T^{-1}L$,	$T^{-1}L$,	$T^{-2}LM$,	$T^{-1}L$,	$T^{-2}LM$.

We see that $Q(s)$ admits an expression of the form (2.12) if we choose the negative of the exponents to T as r_i and c_j, i.e.,

$$r_1 = r_2 = 1,\ r_3 = r_4 = r_5 = 2,\ r_6 = 1;$$

$$c_1 = c_2 = 0,\ c_3 = c_4 = 1,\ c_5 = 2,\ c_6 = 1,\ c_7 = 2.$$

In connection to (2.12) we mention that the following fact is known (see Murota [M1], [M3] for the proof).

Theorem 2.3. *Let $Q(s)$ be an $m \times n$ matrix with entries in $K[s]$, where K ($\supseteq \mathbb{Q}$) is a field and s is an indeterminate over K. Then every nonvanishing subdeterminant of $Q(s)$ is a monomial in s over K if and only if*

$$Q(s) = \mathrm{diag}\,[s^{r_1}, \ldots, s^{r_m}] \cdot Q(1) \cdot \mathrm{diag}\,[s^{-c_1}, \ldots, s^{-c_n}]$$

for some integers r_i ($i = 1, \ldots, m$) and c_j ($j = 1, \ldots, n$).

3 Mathematics on Degree of Determinant

The concept of mixed matrix, introduced in Murota–Iri [MI], is defined formally as follows. Let K be a subfield of a field F. A matrix A over F (i.e., $A_{ij} \in F$) is called a *mixed matrix* with respect to F/K if

$$A = Q + T, \tag{3.1}$$

where

(M-Q) Q is a matrix over K (i.e., $Q_{ij} \in K$), and

(M-T) T is a matrix over F (i.e., $T_{ij} \in F$) such that the set of its nonzero entries is algebraically independent over K.

For example, A_k in (2.4) is a mixed matrix with respect to $F/K = \mathbb{Q}(\mathcal{T})/\mathbb{Q}$, where $\mathbb{Q}(\mathcal{T})$ is the field of rational functions in $\mathcal{T}$ with rational coefficients. See Murota [M7] for a survey on the mathematical properties of mixed matrices.

Similarly, a polynomial matrix $A(s)$ over F (i.e., $A_{ij} \in F(s)$) is called a *mixed polynomial matrix* with respect to F/K if

$$A(s) = Q(s) + T(s) = \sum_{k=0}^{N} s^k Q_k + \sum_{k=0}^{N} s^k T_k \tag{3.2}$$

for some integer $N \geq 0$, where

(MP-Q1) Q_k $(k = 0, 1, \cdots, N)$ are matrices over K, and

(MP-T) T_k $(k = 0, 1, \cdots, N)$ are matrices over F such that the set of their nonzero entries is algebraically independent over K.

A mixed polynomial matrix with respect to F/K is a mixed matrix with respect to $F(s)/K(s)$. It should be obvious that (MP-Q1) and (MP-T) generalize (A-Q1) and (A-T), respectively. Corresponding to (A-Q2) we consider

(MP-Q2) Every nonvanishing subdeterminant of $Q(s)$ is a monomial in s over K.

The following identity is easy to derive from (MP-Q1), (MP-T) and the Laplace expansion of determinants. We denote by R and C the row set and the column set of $A(s)$, respectively.

Theorem 3.1. *For a nonsingular mixed polynomial matrix $A(s) = Q(s) + T(s)$ (satisfying (MP-Q1) and (MP-T)),*

$$\deg \det A = \max_{\substack{|I|=|J| \\ I \subseteq R, J \subseteq C}} \{\deg \det Q[I, J] + \deg \det T[R - I, C - J]\}. \tag{3.3}$$

The right-hand side of this identity involves a maximization over all pairs (I, J), the number of which is $2^{|R|+|C|}$, too large for an exhaustive search for maximization. Fortunately, however, it is possible (Murota [M11]) to design an efficient (polynomial-time) algorithm to compute this maximum. This is based on the fact that the functions $f_Q(I,J) = \deg\det Q[I,J]$ and $f_T(I,J) = \deg\det T[I,J]$ enjoy combinatorially nice properties (abstracted as "valuated bimatroid" in Murota [M9], which is a variant of "valuated matroid" of Dress-Wenzel [DW1,DW2]) and on the recent developments in the theory of valuated matroid. In fact, the algorithm is a straightforward application of a more general algorithmic scheme for the valuated matroid intersection problem developed by Murota [M10]. If the stronger condition (MP-Q2) on $Q(s)$ is satisfied, the computation of the right-hand side of (3.3) is reduced to solving a weighted matroid intersection problem, as has been noted already in Murota [M1, M3].

With this algorithm for the degree of determinant we can compute the index $\nu(A)$ of $A(s)$ based on the formula (1.3): $\nu(A) = \max_{i,j} \deg((i,j)\text{-cofactor of } A) - \deg\det A + 1$. A simplest way is to apply the algorithm repeatedly to the whole matrix A and all the submatrices of order $n-1$ (which are n^2 in number) to obtain $\deg\det A$ and $\deg((i,j)\text{-cofactor of } A)$ for all (i,j). This naive method already gives a polynomial-time algorithm for $\nu(A)$. See Murota [M11] for an improvement on this basic idea.

Let us illustrate the above theorem for the matrix $A^{(1)}(s)$ of (1.1), the first coefficient matrix of our electrical network. We regard it as a mixed polynomial matrix $A^{(1)}(s) = Q^{(1)}(s) + T^{(1)}(s)$ with

$$
Q^{(1)}(s) =
\begin{array}{|ccccc|ccccc|}
\multicolumn{1}{c}{\xi_1} & \xi_2 & \xi_3 & \xi_4 & \multicolumn{1}{c}{\xi_5} & \eta_1 & \eta_2 & \eta_3 & \eta_4 & \multicolumn{1}{c}{\eta_5} \\
\hline
1 & -1 & 0 & 0 & -1 & & & & & \\
-1 & 0 & 1 & 1 & 1 & & & & & \\
\hline
 & & & & & -1 & -1 & 0 & -1 & 0 \\
 & & & & & 0 & 1 & 1 & 0 & -1 \\
 & & & & & 0 & 0 & -1 & 1 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
\hline
\end{array}
,\quad
T^{(1)}(s) =
\begin{array}{|ccccc|ccccc|}
\multicolumn{1}{c}{\xi_1} & \xi_2 & \xi_3 & \xi_4 & \multicolumn{1}{c}{\xi_5} & \eta_1 & \eta_2 & \eta_3 & \eta_4 & \multicolumn{1}{c}{\eta_5} \\
\hline
0 & 0 & 0 & 0 & 0 & & & & & \\
0 & 0 & 0 & 0 & 0 & & & & & \\
\hline
 & & & & & 0 & 0 & 0 & 0 & 0 \\
 & & & & & 0 & 0 & 0 & 0 & 0 \\
 & & & & & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & R_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & R_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & sL & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & sC \\
\hline
\end{array}
$$

under the assumption that R_1, R_2, L and C are independent parameters. The matrix $Q^{(1)}(s)$, being free from s, satisfies the stronger condition (MP-Q2) trivially. On the right-hand side of (3.3) in Theorem 3.1 we can take $I = R - \{\text{row } 8, \text{row } 9\}$, $J = C - \{\text{column } 3, \text{column } 4\}$ as a maximizer, for which

$\deg\det Q^{(1)}[I,J] = 0$ and $\deg\det T^{(1)}[R-I, C-J] = 1$. Hence we obtain $\deg\det A^{(1)} = 1$. In a similar manner we obtain

$$\max_{i,j} \deg\left((i,j)\text{-cofactor of } A^{(1)}\right) = 2,$$

and therefore $\nu(A^{(1)}) = 2-1+1 = 2$ by (1.3). Thus the proposed method gives the correct answer for the matrix $A^{(1)}$ for which the graph-theoretic method fails.

Finally we mention a key fact underlying the efficient algorithm for computing the right-hand side of (3.3), which fact is a special case of the duality theorem for the valuated matroid intersection problem established in Murota [M10] (see Murota [M11] for detail). It asserts the existence of vectors $p_R \in \mathbf{Z}^R$ and $p_C \in \mathbf{Z}^C$ (called "dual variables") that characterize the maximum on the right-hand side of (3.3). Such vectors can be found by the algorithm as a by-product and their physical significance is illustrated below for our electrical network. We use the notation $\mathrm{diag}\,(s;p)$ for a vector $p = (p_i)$ to mean a diagonal matrix having diagonal entries $s^{p_1}, s^{p_2}, \cdots$.

Theorem 3.2. *For a nonsingular mixed polynomial matrix $A(s) = Q(s) + T(s)$ (satisfying (MP-Q1) and (MP-T)), there exist $p_R \in \mathbf{Z}^R$ and $p_C \in \mathbf{Z}^C$ such that $\tilde{A} = \mathrm{diag}\,(s;-p_R)\cdot A \cdot \mathrm{diag}\,(s;p_C) = \tilde{Q} + \tilde{T}$ satisfies*

$$\deg\det\tilde{A} = \max_{\substack{|I|=|J| \\ I\subseteq R, J\subseteq C}} \{\deg\det\tilde{Q}[I,J]\} + \max_{\substack{|I|=|J| \\ I\subseteq R, J\subseteq C}} \{\deg\det\tilde{T}[R-I, C-J]\},$$

where it should be clear that $\tilde{Q} = \mathrm{diag}\,(s;-p_R)\cdot Q \cdot \mathrm{diag}\,(s;p_C)$ and $\tilde{T} = \mathrm{diag}\,(s;-p_R)\cdot T \cdot \mathrm{diag}\,(s;p_C)$.

For the example matrix $A^{(1)}(s)$ of (1.1) we can find, by the algorithm of [M11],

$$p_R = (0,0,0,0,0;0,0,0,0,1), \qquad p_C = (1,0,0,0,1;0,0,0,0,0)$$

as the "dual variables" p_R and p_C in Theorem 3.2 to obtain

$$\tilde{A}^{(1)}(s) = \begin{array}{c} \begin{array}{cccccccccc} Q_1 & \xi_2 & \xi_3 & \xi_4 & Q_5 & \eta_1 & \eta_2 & \eta_3 & \eta_4 & \eta_5 \end{array} \\ \left[\begin{array}{ccccc|ccccc} s & -1 & 0 & 0 & -s & & & & & \\ -s & 0 & 1 & 1 & s & & & & & \\ \hline & & & & & -1 & -1 & 0 & -1 & 0 \\ & & & & & 0 & 1 & 1 & 0 & -1 \\ & & & & & 0 & 0 & -1 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & R_1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & R_2 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & sL & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & C \end{array}\right] \end{array}. \qquad (3.4)$$

We have

$$\deg\det \tilde{A}^{(1)} = 2 = 1 + 1 = \max_{I,J} \deg\det \tilde{Q}^{(1)}[I,J] + \max_{I,J} \deg\det \tilde{T}^{(1)}[I,J]$$

in contrast to

$$\deg\det A^{(1)} = 1 < 0 + 2 = \max_{I,J} \deg\det Q^{(1)}[I,J] + \max_{I,J} \deg\det T^{(1)}[I,J].$$

Considering $\tilde{A} = \mathrm{diag}(s; -p_R) \cdot A \cdot \mathrm{diag}(s; p_C)$ means rewriting $Ax = b$ into $\tilde{A}\tilde{x} = \tilde{b}$ with $\tilde{x} = \mathrm{diag}(s; -p_C) \cdot x$ and $\tilde{b} = \mathrm{diag}(s; -p_R) \cdot b$. In accordance with this observation, the 1st and 5th columns of $\tilde{A}$ are indexed by $Q_1 = s^{-(p_C)_1}\xi_1 = \int \xi_1 dt$ (charge supplied by the source) and $Q_5 = s^{-(p_C)_5}\xi_5 = \int \xi_5 dt$ (charge stored in the capacitor), respectively. The last row represents the constitutive equation $Q_5 = C\eta_5$ rather than $\xi_5 = C\dot{\eta}_5$. It is interesting that the "dual variables" p_R and p_C possess such a physical significance.

4 Conclusion

Polynomial matrices constitute one of the main mathematical tools for the analysis of dynamical systems (Rosenbrock [R2], Vidyasagar [V]) and the method of structural analysis based on mixed polynomial matrices is not restricted to the index problem. In fact, other problems such as controllability (Murota [M2]), fixed mode (Murota [M5]), disturbance decoupling (Murota–van der Woude [MW]) have been treated in this framework. See Murota [M4] for a survey.

The author is indebted to Francois Cellier and Pawel Bujakiewicz for the problem of electrical network and to Masaaki Sugihara and Satoru Iwata for suggestions to improve the presentation.

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