Handout for Summer 2005

Mixed Matrices
— Irreducibility and Decomposition — [*]

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Abstract

This paper surveys mathematical properties of (layered-) mixed matrices with emphasis on irreducibility and block-triangular decomposition. A matrix A is a mixed matrix if $A = Q + T$, where Q is a "constant" matrix and T is a "generic" matrix (or formal incidence matrix) in the sense that the nonzero entries of T are algebraically independent parameters. A layered mixed (or LM-) matrix is a mixed matrix such that Q and T have disjoint nonzero rows, i.e., no row of $A = Q + T$ has both a nonzero entry from Q and a nonzero entry from T. The irreducibility for an LM-matrix is defined with respect to a natural admissible transformation as an extension of the well-known concept of full indecomposability for a generic matrix. Major results for fully indecomposable generic matrices such as Frobenius' characterization in terms of the irreducibility of determinant are generalized. As for block-triangularization, the Dulmage-Mendelsohn decomposition is generalized to the combinatorial canonical form (CCF) of an LM-matrix along with the uniqueness and the algorithm. Matroid-theoretic methods are useful for investigating a mixed matrix.

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[*] *Combinatorial and Graph-Theoretic Problems in Linear Algebra* (eds.: R. A. Brualdi, S. Friedland and V. Klee), The IMA Volumes in Mathematics and Its Applications, Vol. 50, Springer, 1993, pp.39–71. See also: K. Murota: Matrices and Matroids for Systems Analysis, Algorithms and Combinatorics, Vol.20, Springer-Verlag, 2000 (ISBN=3-54066-024-0).

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1 Introduction

The notion of a mixed matrix was introduced by Murota-Iri [40] as a mathematical tool for systems analysis by means of matroid-theoretic combinatorial methods. A matrix A is called a mixed matrix if $A = Q + T$, where Q is a "constant" matrix and T is a "generic" matrix (or formal incidence matrix) in the sense that the nonzero entries of T are algebraically independent [52] parameters (see below for the precise definition). A layered mixed (or LM-) matrix is defined (see below) as a mixed matrix such that Q and T have disjoint nonzero rows, i.e., no row of $A = Q + T$ has both a nonzero entry from Q and a nonzero entry from T.

The notion of a mixed matrix is motivated by the following physical observation. When we describe a physical system (such as an electrical network, a chemical plant) in terms of elementary variables, we can often distinguish following two kinds of numbers, together characterizing the physical system.

Inaccurate Numbers: Numbers representing independent physical parameters such as masses in mechanical systems and resistances in electrical networks. Such numbers are contaminated with noise and other errors and take values independent of one another; therefore they can be modeled as algebraically independent numbers, and

Accurate Numbers: Numbers accounting for various sorts of conservation laws such as Kirchhoff's laws. Such numbers stem from topological incidence relations and are precise in value (often ± 1); therefore they cause no serious numerical difficulty in arithmetic operations on them.

The "inaccurate numbers" constitute the matrix T whereas the "accurate numbers" the matrix Q. We may also refer to the numbers of the first kind as "system parameters" and to those of the second kind as "fixed constants". In this paper we do not discuss physical/engineering significance of a mixed matrix, but concentrate on its mathematical properties. See [25], [27], [28], [30], [31], [37], [38], [39], [40], [42] for engineering applications of mixed matrices; and Chen [6], Iri [18], Recski [46], Yamada-Foulds [56] for graph/matroid theoretic methods for systems analysis.

Here is a preview of some nice properties enjoyed by a mixed matrix or an LM-matrix.

- The rank is expressed as the minimum of a submodular function (Theorem 5) and can be computed efficiently by a matroid-theoretic algorithm.

- A notion of irreducibility is defined with respect to a natural transformation of physical significance. The irreducibility for an LM-matrix is an extension of the well-known concept of full indecomposability for a generic matrix.

- An irreducible component thus defined satisfies a number of nice properties that justify the name of irreducibility (Theorems 10, 11, 12, 13). Many results for a fully indecomposable generic matrix are extended, including Frobenius' characterization in terms of the irreducibility of determinant.

- There exists a unique canonical block-triangular decomposition, called the combinatorial canonical form (CCF for short), into irreducible components (Theorem 6). This is a generalization of the Dulmage-Mendelsohn decomposition. The CCF can be computed by an efficient algorithm (see §6).

We now give the precise definitions of mixed matrix, layered mixed matrix and admissible transformation for a layered mixed matrix. For a matrix A, the row set and the column set of A are denoted by $\mathrm{Row}(A)$ and $\mathrm{Col}(A)$. For $I \subseteq \mathrm{Row}(A)$ and $J \subseteq \mathrm{Col}(A)$, $A[I, J] = (A_{ij} \mid i \in I, j \in J)$ means the submatrix of A with row set I and column set J. The rank of A is written as $\mathrm{rank}\, A$.

Let $\boldsymbol{K}$ be a subfield of a field $\boldsymbol{F}$. An $m \times n$ matrix A over $\boldsymbol{F}$ (i.e., $A_{ij} \in \boldsymbol{F}$) is called a *mixed matrix* with respect to $\boldsymbol{F}/\boldsymbol{K}$ if

$$A = Q + T, \tag{1}$$

where

(M1) Q is an $m \times n$ matrix over $\boldsymbol{K}$ (i.e., $Q_{ij} \in \boldsymbol{K}$), and

(M2) T is an $m \times n$ matrix over $\boldsymbol{F}$ (i.e., $T_{ij} \in \boldsymbol{F}$) such that the set of its nonzero entries is algebraically independent [52] over $\boldsymbol{K}$.

The subfield $\boldsymbol{K}$ will be called the *base field*.

A mixed matrix A of (1) is called a *layered mixed matrix* (or an *LM-matrix*) with respect to $\boldsymbol{F}/\boldsymbol{K}$ if the nonzero rows of Q and T are disjoint. In other words, A is an LM-matrix, denoted as $A \in \mathrm{LM}(\boldsymbol{F}/\boldsymbol{K}) = \mathrm{LM}(\boldsymbol{F}/\boldsymbol{K}; m_Q, m_T, n)$, if it can be put into the following form with a permutation of rows:

$$A = \begin{pmatrix} Q \\ T \end{pmatrix} = \begin{pmatrix} Q \\ O \end{pmatrix} + \begin{pmatrix} O \\ T \end{pmatrix}, \tag{2}$$

where

(L1) Q is an $m_Q \times n$ matrix over $\boldsymbol{K}$ (i.e., $Q_{ij} \in \boldsymbol{K}$), and

(L2) T is an $m_T \times n$ matrix over $\boldsymbol{F}$ (i.e., $T_{ij} \in \boldsymbol{F}$) such that the set $\mathcal{T}$ of its nonzero entries is algebraically independent over $\boldsymbol{K}$.

Though an LM-matrix is, by definition, a special case of mixed matrix, the following argument would indicate that the class of LM-matrices is as general as the class of mixed matrices both in theory and in application. Consider a system of equations $A\boldsymbol{x} = \boldsymbol{b}$ described with an $m \times n$ mixed matrix $A = Q + T$. By introducing an auxiliary variable $\boldsymbol{w} \in \boldsymbol{F}^m$ we can rewrite the equation as

$$\tilde{A} \begin{pmatrix} \boldsymbol{w} \\ \boldsymbol{x} \end{pmatrix} = \begin{pmatrix} \boldsymbol{b} \\ \boldsymbol{0} \end{pmatrix}$$

with a $(2m) \times (m + n)$ LM-matrix

$$\tilde{A} = \begin{pmatrix} \tilde{Q} \\ \tilde{T} \end{pmatrix} = \begin{pmatrix} I_m & Q \\ -\mathrm{diag}\,[t_1, \ldots, t_m] & T' \end{pmatrix}, \tag{3}$$

where $\mathrm{diag}\,[t_1, \ldots, t_m]$ is a diagonal matrix with "new" variables $t_1, \ldots, t_m (\in \boldsymbol{F})$, and $T'_{ij} = t_i T_{ij}$. Note that $\mathrm{rank}\, \tilde{A} = \mathrm{rank}\, A + m$.

Example 1 An equation described with a mixed matrix:

$$\begin{pmatrix} 2+\alpha & 3 \\ \beta & 4+\gamma \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix},$$

where $\mathcal{T} = \{\alpha, \beta, \gamma\}$ is algebraically independent, can be rewritten as

$$\begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & 4 \\ -t_1 & 0 & t_1\alpha & 0 \\ 0 & -t_2 & t_2\beta & t_2\gamma \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ 0 \\ 0 \end{pmatrix}$$

by means of an LM-matrix. $\Box$

For an LM-matrix $A \in \mathrm{LM}(\boldsymbol{F}/\boldsymbol{K}\,; m_Q, m_T, n)$ of (2) we define an *admissible transformation* to be a transformation of the form:

$$P_\mathrm{r} \begin{pmatrix} S & O \\ O & I \end{pmatrix} \begin{pmatrix} Q \\ T \end{pmatrix} P_\mathrm{c}, \tag{4}$$

where P_r and P_c are permutation matrices, and S is a nonsingular matrix over the base field $\boldsymbol{K}$ (i.e., $S \in \mathrm{GL}(m_Q, \boldsymbol{K})$).

An admissible transformation brings an LM-matrix into another LM-matrix and two LM-matrices are said to be *LM-equivalent* if they are connected by an admissible transformation. If A' is LM-equivalent to A, then $\mathrm{Col}(A')$ may be identified with $\mathrm{Col}(A)$ through the permutation P_c. Examples 2, 3 below illustrate the admissible transformation.

With respect to the admissible transformation (4) we can define the notion of irreducibility for LM-matrices, which is an extension of the well-studied concept of full indecomposability [5], [49]. First recall that a matrix A' is said to be *partially decomposable* if it contains a zero submatrix $A'[I, J] = O$ with $|I| + |J| = \max(|\mathrm{Row}(A')|, |\mathrm{Col}(A')|)$; otherwise, it is called *fully indecomposable*. An LM-matrix $A \in \mathrm{LM}(\boldsymbol{F}/\boldsymbol{K}\,; m_Q, m_T, n)$ is defined to be *LM-reducible* if it can be decomposed into smaller submatrices by means of the admissible transformation, or more precisely, if there exists a partially decomposable matrix A' which is LM-equivalent to A. On the other hand, A will be called *LM-irreducible* if it is not LM-reducible, that is, if any LM-matrix A' equivalent to A is fully indecomposable. Hence, if A is LM-irreducible, then it is fully indecomposable; but not conversely. By convention A is regarded as LM-irreducible if $\mathrm{Row}(A) = \emptyset$ or $\mathrm{Col}(A) = \emptyset$.

Let us consider the special case where $m_Q = 0$. Then $A = T$ and hence all the nonzero entries are algebraically independent. Such a matrix is called a *generic matrix* in Brualdi-Ryser [5]. The admissible transformation (4) reduces to $\bar{A} = P_\mathrm{r} A P_\mathrm{c}$, involving permutations only, and the LM-irreducibility is nothing but the full indecomposability. It is known that a fully indecomposable generic matrix enjoys a number of interesting properties. On the other hand, if a matrix is not fully indecomposable, it can be decomposed uniquely into fully indecomposable components. This is called the *Dulmage-Mendelsohn decomposition*, or the *DM-decomposition* for short. See [4], [5], [8], [21], [28], [33], [37] for more about the DM-decomposition.

In this paper we are mainly interested in whether these results for a generic matrix can be extended to a general LM-matrix. It will be shown that many major results for a fully indecomposable generic matrix are extended for an LM-irreducible matrix, and the DM-decomposition is extended to a canonical block-triangular decomposition under the admissible transformation (4). The canonical form is called the combinatorial canonical form (CCF) of an LM-matrix, which is illustrated in the following examples, whereas a precise description of the CCF will be given as Theorem 6 in §3.

Example 2 Consider a 3×3 LM-matrix

$$A = \begin{pmatrix} Q \\ T \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 3 \\ 0 & t_1 & t_2 \end{pmatrix}$$

with

$$Q = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 3 \end{pmatrix}, \quad T = (\, 0 \quad t_1 \quad t_2 \,),$$

where $\mathcal{T} = \{t_1, t_2\}$ is the set of algebraically independent parameters. This matrix is fully indecomposable (DM-irreducible) and cannot be decomposed into smaller blocks by means of permutations of rows and columns. By choosing $S = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ and $P_{\rm r} = P_{\rm c} = I$ in the admissible transformation (4), we can obtain a block-triangular decomposition:

$$\bar{A} = \begin{pmatrix} SQ \\ T \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ & 1 & 3 \\ & t_1 & t_2 \end{pmatrix}.$$

Thus the admissible transformation is more powerful than mere permutations. □

Example 3 Consider an LM-matrix $A = \begin{pmatrix} Q \\ T \end{pmatrix}$ of (2) defined by

$$Q = \left(\begin{array}{cccccccccc} \xi_1 & \xi_2 & \xi_3 & \xi_4 & \xi_5 & \eta_1 & \eta_2 & \eta_3 & \eta_4 & \eta_5 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right),$$

$$T = \left(\begin{array}{cccccccccc} \xi_1 & \xi_2 & \xi_3 & \xi_4 & \xi_5 & \eta_1 & \eta_2 & \eta_3 & \eta_4 & \eta_5 \\ r_1 & 0 & 0 & 0 & 0 & t_1 & 0 & 0 & 0 & 0 \\ 0 & r_2 & 0 & 0 & 0 & 0 & t_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha & 0 & t_3 & 0 & 0 \\ 0 & \beta & 0 & t_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_5 \end{array}\right),$$

where $\mathcal{T} = \{r_1, r_2, \alpha, \beta; t_1, \cdots, t_5\}$ is the set of algebraically independent parameters. (See Example 16.2 in [28] for the physical meaning of this example.)

The combinatorial canonical form (CCF), i.e., the finest block-triangular form under the admissible transformation (4) is obtained as follows. Choosing

$$S = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

in (4) we first transform Q to

$$Q' = SQ = \begin{array}{c} \begin{array}{cccccccccc} \xi_1 & \xi_2 & \xi_3 & \xi_4 & \xi_5 & \eta_1 & \eta_2 & \eta_3 & \eta_4 & \eta_5 \end{array} \\ \begin{pmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & -1 \end{pmatrix} \end{array},$$

and then permute the rows and the columns of $\begin{pmatrix} Q' \\ T \end{pmatrix}$ with permutation matrices

$$P_{\mathrm{r}} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$P_{\mathrm{c}} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

to obtain an explicit block-triangular LM-matrix

$$\bar{A} = P_{\mathrm{r}} \begin{pmatrix} Q' \\ T \end{pmatrix} P_{\mathrm{c}} = \begin{array}{c} \begin{array}{cccccccccc} \xi_3 & \xi_5 & \eta_4 & \xi_1 & \xi_2 & \xi_4 & \eta_1 & \eta_2 & \eta_3 & \eta_5 \end{array} \\ \begin{pmatrix} 1 & & & & -1 & & & & & \\ & 1 & & -1 & & & & & & \\ & & 1 & & & & -1 & & & -1 \\ & & & 1 & 1 & 1 & 0 & 0 & 0 & \\ & & & 0 & 0 & 0 & -1 & 1 & 1 & -1 \\ & & & r_1 & 0 & 0 & t_1 & 0 & 0 & \\ & & & 0 & r_2 & 0 & 0 & t_2 & 0 & \\ & & & 0 & 0 & 0 & \alpha & 0 & t_3 & \\ & & & 0 & \beta & t_4 & 0 & 0 & 0 & \\ & & & & & & & & & t_5 \end{pmatrix} \end{array}.$$

This is the CCF of A, namely, the finest block-triangular matrix which is LM-equivalent to A. Hence A is LM-reducible whereas each diagonal block of $\bar{A}$ is LM-irreducible. The columns of $\bar{A}$ are partitioned into five blocks:

$$C_1 = \{\xi_3\},\ C_2 = \{\xi_5\},\ C_3 = \{\eta_4\},\ C_4 = \{\xi_1, \xi_2, \xi_4, \eta_1, \eta_2, \eta_3\},\ C_5 = \{\eta_5\}.$$

The zero/nonzero structure of $\bar{A}$ determines the following partial order among the blocks:

$$\begin{tikzcd}
 & C_5 & \\
 & C_4 \arrow[u] & \\
C_1 \arrow[ur] & C_2 \arrow[u] & C_3 \arrow[ul]
\end{tikzcd}$$

This partial order indicates, for example, that the blocks C_1 and C_2, having no order relation, could be exchanged in position without destroying the block-triangular form provided the corresponding rows are exchanged in position accordingly. This corresponds to the fact that the entry in the first row of the column ξ_5 is equal to 0. A precise description of the CCF and a combinatorial characterization of the partial order will be given in Theorem 6 in §3. The transformation matrices S, P_{r} and P_{c} can be found by the algorithm described in §6. $\square$

2 Rank

In this section we consider combinatorial characterizations of the rank of an LM-matrix $A = \binom{Q}{T} \in \mathrm{LM}(\boldsymbol{F}/\boldsymbol{K})$. We put $C = \mathrm{Col}(A)$, $R = \mathrm{Row}(A)$, $R_Q = \mathrm{Row}(Q)$ and $R_T = \mathrm{Row}(T)$; then $\mathrm{Col}(Q) = \mathrm{Col}(T) = C$, and $R = R_Q \cup R_T$.

Before dealing with a general LM-matrix, let us consider the special case of a generic matrix, i.e., where $A = T$ (with $m_Q = 0$) and hence all the nonzero entries are algebraically independent over $\boldsymbol{K}$. The zero/nonzero structure of T can be conveniently represented by a bipartite graph $G(T) = (\mathrm{Row}(T), \mathrm{Col}(T), \mathcal{T})$, which has $\mathrm{Row}(T) \cup \mathrm{Col}(T)$ as the vertex set and $\mathcal{T}$ (=set of nonzero entries of T) as the arc set. The *term-rank* of T, denoted as term-rank T, is equal to the maximum size of a matching in $G(T)$. In other words, term-rank T is the maximum size of a square submatrix $T[I, J]$ such that there exists a one-to-one correspondence $\pi : I \to J$ with $T_{i\pi(i)} \neq 0$ $(\forall i \in I)$:

$$\text{term-rank}\, T = \max\{|I| \mid \exists\, \pi(\text{one-to-one}) : I \to J,\ \forall i \in I : T_{i\pi(i)} \neq 0\}.$$

The following fact is well known [5], [10]. See Lemma 3 below for the proof.

Lemma 1 *For a generic matrix T, which has algebraically independent nonzero entries, we have*

$$\mathrm{rank}\, T = \text{term-rank}\, T.$$

□

The zero/nonzero structure of T is represented by the functions τ, $\gamma : 2^{R_T} \times 2^C \to \mathbf{Z}$ defined as

$$\begin{aligned}
\tau(I, J) &= \text{term-rank}\, T[I, J], \quad I \subseteq R_T, J \subseteq C, && (5)\\
\Gamma(I, J) &= \bigcup_{j \in J} \{i \in I \mid T_{ij} \neq 0\}, \quad I \subseteq R_T, J \subseteq C, && (6)\\
\gamma(I, J) &= |\Gamma(I, J)|, \quad I \subseteq R_T, J \subseteq C. && (7)
\end{aligned}$$

Lemma 1 shows that $\tau(I, J) = \mathrm{rank}\, T[I, J]$, whereas $\Gamma(I, J)$ stands for the set of nonzero rows of the submatrix $T[I, J]$, and $\gamma(I, J)$ for the number of nonzero rows of $T[I, J]$.

These functions τ, γ enjoy *bisubmodularity*, that is, they each satisfy an inequality of the following type:

$$f(I_1 \cup I_2, J_1 \cap J_2) + f(I_1 \cap I_2, J_1 \cup J_2) \leq f(I_1, J_1) + f(I_2, J_2). \tag{8}$$

For a bisubmodular function f in general, $f_I \equiv f(I, \cdot) : 2^C \to \mathbf{Z}$, for each I, is a *submodular* function:

$$f_I(J_1 \cap J_2) + f_I(J_1 \cup J_2) \leq f_I(J_1) + f_I(J_2),\ J_i \subseteq C\ (i = 1, 2). \tag{9}$$

The following fact is a version of the fundamental minimax relation concerning the maximum matchings and the minimum covers of a bipartite graph, which is often associated with J. Egerváry, G. Frobenius, D. König, P. Hall, R. Rado, O. Ore, and others [5], [20], [21], [53]. Note also that the function $\gamma(I, J) - |J|$ (with I fixed) is called the surplus function in Lovász-Plummer [21].

Lemma 2 *For τ and γ defined by (5) and (7),*

$$\tau(I, J) = \min\{\gamma(I, J') - |J'| \mid J' \subseteq J\} + |J|, \quad I \subseteq R_T, J \subseteq C.$$

□

We are now in the position to consider the rank of a general LM-matrix. The following lemma is a fundamental identity for an LM-matrix, an extension of Lemma 1 for a generic matrix. It will be translated first into a matroid-theoretic expression in Lemma 4, and then, with the aid of the matroid partition theorem, turned into the important minimax formulas in Theorem 5.

Lemma 3 *For $A \in \mathrm{LM}(\boldsymbol{F}/\boldsymbol{K})$,*

$$\mathrm{rank}\, A = \max\{\mathrm{rank}\, Q[R_Q, J] + \text{term-rank}\, T[R_T, C - J] \mid J \subseteq C\}. \tag{10}$$

(Proof) First assume that A is square and consider the (generalized) Laplace expansion [16]:

$$\det A = \sum_{J \subseteq C} \pm \det Q[R_Q, J] \cdot \det T[R_T, C - J].$$

If $\det A \neq 0$, then both $Q[R_Q, J]$ and $T[R_T, C - J]$ are nonsingular for some J. The algebraic independence of $\mathcal{T}$ ensures the converse. This shows (10) for a square A. For a nonsquare matrix A, the same argument applies to its square submatrices. □

For the matrix A of Example 3, we may take $J = \{\xi_5, \xi_3, \xi_4, \eta_4, \eta_3\}$ for the subset that attains the maximum (=10) on the right-hand side of (10). Therefore A is nonsingular.

Let us introduce some matroid-theoretic concepts to recast the identity in Lemma 3. Put $\mathcal{F} = \{J \subseteq C \mid \mathrm{rank}\, A[R, J] = |J|\}$, which denotes the family of linearly independent columns of A. As is easily verified, the family $\mathcal{F}$ satisfies the following three conditions:

(i) $\emptyset \in \mathcal{F}$,

(ii) $J_1 \subseteq J_2 \in \mathcal{F} \Rightarrow J_1 \in \mathcal{F}$,

(iii) $J_1 \in \mathcal{F}$, $J_2 \in \mathcal{F}$, $|J_1| < |J_2| \Rightarrow J_1 \cup \{j\} \in \mathcal{F}$ for some $j \in J_2 - J_1$.

In general, a pair $\boldsymbol{M} = (C, \mathcal{F})$ of a finite set C and a family $\mathcal{F}$ of subsets of C is called a *matroid* if it satisfies the three conditions above. C is called the *ground set* and $\mathcal{F}$ the *family of independent sets*. A maximal member (with respect to set inclusion) in $\mathcal{F}$ is called a *base*, and, by condition (ii), $\mathcal{F}$ is determined by the family $\mathcal{B}$ of bases. The size of a base is uniquely determined, which is called the *rank* of $\boldsymbol{M}$, denoted as $\mathrm{rank}\, \boldsymbol{M}$; i.e., $\mathrm{rank}\, \boldsymbol{M} = |B| = \max\{|J| \mid J \in \mathcal{F}\}$ for $B \in \mathcal{B}$. Given two matroids $\boldsymbol{M}_1 = (C, \mathcal{F}_1)$ and $\boldsymbol{M}_2 = (C, \mathcal{F}_2)$ with the same ground set C, another matroid, denoted as $(C, \mathcal{F}_1 \vee \mathcal{F}_2)$, is defined by

$$\mathcal{F}_1 \vee \mathcal{F}_2 = \{J_1 \cup J_2 \mid J_1 \in \mathcal{F}_1,\ J_2 \in \mathcal{F}_2\}.$$

This is called the *union* of $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$, and denoted as $\boldsymbol{M}_1 \vee \boldsymbol{M}_2$. See [15], [20], [53], [54], [55] for more about matroids.

For an LM-matrix $A = \binom{Q}{T} \in \mathrm{LM}(\boldsymbol{F}/\boldsymbol{K})$ we consider the matroids $\boldsymbol{M}(A)$, $\boldsymbol{M}(Q)$, $\boldsymbol{M}(T)$ on C defined respectively by matrices A, Q, T with respect to the linear independence among column vectors. Then Lemma 3 is rewritten as follows.

Lemma 4 *For* $A = \binom{Q}{T} \in \mathrm{LM}(\boldsymbol{F}/\boldsymbol{K})$, *we have* $\boldsymbol{M}(A) = \boldsymbol{M}(Q) \vee \boldsymbol{M}(T)$. □

This theorem makes it possible to compute the rank of an LM-matrix with $\mathrm{O}(n^3 \log n)$ arithmetic operations (assuming $m = \mathrm{O}(n)$ for simplicity) in the base field $\boldsymbol{K}$ by utilizing an established algorithm for matroid partition/union problem ([7], [9], [11], [15], [20], [53], [54], [55]). See the algorithm in §6.

As an extension of the surplus function for a generic matrix (cf. Lemma 2) we introduce a set function $p : 2^R \times 2^C \to \mathbf{Z}$ as follows. For $A \in \mathrm{LM}(\boldsymbol{F}/\boldsymbol{K})$ we define p by

$$p(I, J) = \rho(I \cap R_Q, J) + \gamma(I \cap R_T, J) - |J|, \quad I \subseteq R, J \subseteq C, \tag{11}$$

where

$$\rho(I, J) = \operatorname{rank} Q[I, J], \quad I \subseteq R_Q, J \subseteq C, \tag{12}$$

stands for the "constant" matrix Q, whereas γ (see (7) for the definition) represents the combinatorial structure of T. Note that, in the special case where $A = T$ (i.e., $m_Q = 0$), we have $p(I, J) = \gamma(I, J) - |J|$, which is the surplus function used in Lemma 2.

The function p is bisubmodular (cf. (8)) and therefore $p_I \equiv p(I, \cdot) : 2^C \to \mathbf{Z}$ is submodular for each $I \subseteq R$, namely,

$$p_I(J_1 \cap J_2) + p_I(J_1 \cup J_2) \leq p_I(J_1) + p_I(J_2), \quad J_i \subseteq C \ (i = 1, 2). \tag{13}$$

The submodular function p_R (i.e., p_I with $I = R$) is invariant under the LM-equivalence in the sense that, if A' is LM-equivalent to A, then $\mathrm{Col}(A')$ may be identified with $C = \mathrm{Col}(A)$ and the functions p and p' associated respectively with A and A' satisfy $p(\mathrm{Row}(A), J) = p'(\mathrm{Row}(A'), J)$ for $J \subseteq C$.

The following theorem gives two minimax expressions (14) and (15), similar but different, for the rank of an LM-matrix. The second expression (15) (or equivalently (16)), due to Murota [26] [28], Murota-Iri-Nakamura [41], is an extension of the minimax relation between matchings and covers given in Lemma 2. In fact, the expression (15) with $\rho = 0$ reduces to Lemma 2 since then $\operatorname{rank} A[I, J] = \operatorname{rank} T[I, J] = \tau(I, J)$ by Lemma 1.

Theorem 5 *Let* $A \in \mathrm{LM}(\boldsymbol{F}/\boldsymbol{K})$ *and* $I \subseteq R$, $J \subseteq C$. *Then*

$$\operatorname{rank} A[I, J] = \min\{\rho(I \cap R_Q, J') + \tau(I \cap R_T, J') - |J'| \mid J' \subseteq J\} + |J|, \tag{14}$$

$$\operatorname{rank} A[I, J] = \min\{\rho(I \cap R_Q, J') + \gamma(I \cap R_T, J') - |J'| \mid J' \subseteq J\} + |J|. \tag{15}$$

Using the function p_R *the latter formula for* $I = R$, $J = C$ *can be written as*

$$\operatorname{rank} A = \min\{p_R(J) \mid J \subseteq C\} + |C|. \tag{16}$$

(Proof) Lemma 4 shows that $\operatorname{rank} A = \operatorname{rank} \boldsymbol{M}(A) = \operatorname{rank} (\boldsymbol{M}(Q) \vee \boldsymbol{M}(T))$. On the other hand, the matroid union/partition theorem of Edmonds [9] (see also [11], [15], [20], [53], [54], [55]) says that

$$\operatorname{rank} (\boldsymbol{M}(Q) \vee \boldsymbol{M}(T)) = \min\{\operatorname{rank} Q(R_Q, J) + \operatorname{rank} T(R_T, J) + |C - J| \mid J \subseteq C\},$$

which establishes (14) for $I = R$, $J = C$. The same argument applied to the submatrix $A[I, J]$ shows (14).

The right-hand sides of (14) and (15) are equal, since with the notations $I \cap R_Q = I_Q$, $I \cap R_T = I_T$, we have

$$\begin{aligned}
&\min_{J' \subseteq J} \{\rho(I_Q, J') + \tau(I_T, J') - |J'|\} \\
&= \min_{J' \subseteq J} \{\rho(I_Q, J') + \min_{J'' \subseteq J'} \{\gamma(I_T, J'') - |J''|\}\} \\
&= \min_{J'' \subseteq J} \{ \min_{J' \supseteq J''} \{\rho(I_Q, J')\} + \gamma(I_T, J'') - |J''|\} \\
&= \min_{J'' \subseteq J} \{\rho(I_Q, J'') + \gamma(I_T, J'') - |J''|\},
\end{aligned}$$

where the first equality is by Lemma 2 and the last equality is due to the monotonicity of $\rho(I_Q, J)$ with respect to J for a fixed I_Q. □

The two expressions in Theorem 5 look very similar, with τ in (14) replaced by γ in (15). Moreover, in both formulas, the functions to be minimized are submodular in J'. However, we will see in the next section that the second expression (15), not the first one, chimes in exact harmony with the admissible transformation (4), with respect to which we are to consider the block-triangular decomposition.

3 Decomposition (CCF)

3.1 Description of CCF

This section gives a precise description, Theorem 6 below, of the combinatorial canonical form (CCF), which has already been sketched informally in Examples 2, 3 in Introduction.

As stated in Theorem 5, the rank of $A[I, J]$ is expressed by the minimum of p_I. Then it would be natural to look at the family of minimizers:

$$L(p_I) = \{J \subseteq C \mid p(I, J) \leq p(I, J'), \forall J' \subseteq C\}, \quad I \subseteq R, \tag{17}$$

which, for each $I \subseteq R$, forms a sublattice of 2^C by virtue of the submodularity (13) of p_I. In fact, if both J_1 and J_2 attain the minimum value, say α, of p_I, then $2\alpha \leq p_I(J_1 \cap J_2) + p_I(J_1 \cup J_2) \leq p_I(J_1) + p_I(J_2) = 2\alpha$ shows that $J_1 \cap J_2 \in L(p_I)$ and $J_1 \cup J_2 \in L(p_I)$. The sublattice $L(p_R)$ plays a crucial role for the block-triangular decomposition, as explained below.

Here we make use of some fundamental results from lattice theory [2], [3]. Birkhoff's representation theorem implies that there exists a one-to-one correspondence between sublattices of 2^C and pairs of a partition of C into blocks and a partial order among the blocks. This correspondence is given as follows.

Let L be a sublattice of 2^C. Take any maximal ascending chain:

$$X_0 \ (= \min L) \subset X_1 \subset \cdots \subset X_b \ (= \max L),$$

where $X_k \in L$, and put

$$\begin{aligned}
C_0 &= X_0, \\
C_k &= X_k - X_{k-1} \quad (k = 1, \ldots, b), \\
C_\infty &= C - X_b.
\end{aligned} \tag{18}$$

Then the family of the subsets $\{C_k \mid k = 1, \ldots, b\}$ is uniquely determined, being independent of the choice of the chain. A partial order $\preceq$ is introduced on $\{C_k \mid k = 1, \ldots, b\}$ by

$$C_k \preceq C_l \quad \Longleftrightarrow \quad [X \in L,\ C_l \subseteq X \ \Rightarrow \ C_k \subseteq X].$$

For convenience, we extend the partial order onto

$$\{C_0, C_\infty\} \cup \{C_k \mid k = 1, \ldots, b\}$$

by defining

$$\begin{aligned} C_0 &\preceq C_k \quad (k = 1, \ldots, b) \quad \text{if} \quad C_0 \neq \emptyset, \\ C_k &\preceq C_\infty \quad (k = 1, \ldots, b) \quad \text{if} \quad C_\infty \neq \emptyset. \end{aligned}$$

We also introduce the following notation:

$$\begin{aligned} C_k \prec C_l \quad &\Longleftrightarrow \quad C_k \preceq C_l \ \text{ and } C_k \neq C_l; \\ C_k \prec\cdot\, C_l \quad &\Longleftrightarrow \quad \begin{cases} \text{(i) } C_k \prec C_l \quad \text{and} \\ \text{(ii) } \not\exists\, C_j \text{ such that } \ C_k \prec C_j \prec C_l. \end{cases} \end{aligned}$$

In this way, a sublattice L of 2^C determines a pair of a partition $\{C_0; C_1, \ldots, C_b; C_\infty\}$ and a partial order $\preceq$, which we denote by

$$\mathcal{P}(L) = (\{C_0; C_1, \ldots, C_b; C_\infty\}, \preceq). \tag{19}$$

Note that $C_k \neq \emptyset$ for $k = 1, \ldots, b$, whereas C_0 and C_∞ are distinguished blocks that can be empty. It may also be mentioned that a pair of a partition of C and a partial order among the blocks is nothing but a quasi-order (=reflexive and transitive binary relation [2]) on C.

Conversely, given $\mathcal{P} = (\{C_0; C_1, \ldots, C_b; C_\infty\}, \preceq)$, a sublattice L is determined as follows: $X \in L$ if and only if $C_0 \subseteq X \subseteq C - C_\infty$ and, for $1 \leq l \leq b$,

$$X \cap C_l \neq \emptyset \quad \Rightarrow \quad \bigcup_{C_k \preceq C_l} C_k \subseteq X.$$

Namely, L is the family of (order-) ideals containing C_0 and contained in $C - C_\infty$. Note that $\min L = C_0$ and $\max L = C - C_\infty$. This correspondence between L and $\mathcal{P}$ is known to be a one-to-one correspondence.

According to this general principle, the sublattice $L(p_R)$ associated with an LM-matrix A determines $\mathcal{P}(L(p_R))$, a pair of a partition of C and a partial order $\preceq$. Note that by (18) the blocks are indexed consistently with the partial order in the sense that

$$C_k \preceq C_l \quad \Rightarrow \quad k \leq l. \tag{20}$$

The following theorem, established in an unpublished report by Murota [26] in 1985 and published as Murota [28], Murota-Iri-Nakamura [41], claims the existence of the CCF of an LM-matrix. The construction of CCF is described in the next subsection along with an outline of the proof. A complete proof can be found in [26], [28], [41].

Theorem 6 *For an LM-matrix* $A \in \mathrm{LM}(\boldsymbol{F}/\boldsymbol{K})$ *there exists another LM-matrix* $\bar{A}$ *which is LM-equivalent to* A *and satisfies the following properties.*
(B1) [Nonzero structure and partial order $\preceq$] $\bar{A}$ *is block-triangularized, i.e.,*

$$\bar{A}[R_k, C_l] = O \quad \text{if} \quad 0 \leq l < k \leq \infty,$$

where $\{R_0; R_1, \ldots, R_b; R_\infty\}$ *and* $\{C_0; C_1, \ldots, C_b; C_\infty\}$ *are partitions of* $\mathrm{Row}(\bar{A})$ *and* $\mathrm{Col}(\bar{A})$ *respectively such that* $R_k \neq \emptyset$, $C_k \neq \emptyset$ *for* $k = 1, \ldots, b$, *whereas* R_0, R_∞, C_0 *and* C_∞ *can be empty.*

Moreover, when $\mathrm{Col}(\bar{A})$ *is identified with* $\mathrm{Col}(A)$, *the partition* $\{C_0; C_1, \ldots, C_b;\, C_\infty\}$ *agrees with that defined by the lattice* $L(p_R)$ *and the partial order on* $\{C_1, \ldots, C_b\}$ *induced by the zero/nonzero structure of* $\bar{A}$ *agrees with the partial order* $\preceq$ *defined by* $L(p_R)$; *i.e.,*

$$\begin{aligned} \bar{A}[R_k, C_l] &= O \quad \text{unless} \quad C_k \preceq C_l \quad (1 \leq k, l \leq b); \\ \bar{A}[R_k, C_l] &\neq O \quad \text{if} \quad C_k \prec\cdot\, C_l \quad (1 \leq k, l \leq b). \end{aligned}$$

(B2) [Size of the diagonal blocks]

$$\begin{aligned} |R_0| &< |C_0| \quad \text{if} \quad R_0 \neq \emptyset, \\ |R_k| &= |C_k| \quad (>0) \quad \text{for} \quad k = 1, \ldots, b, \\ |R_\infty| &> |C_\infty| \quad \text{if} \quad C_\infty \neq \emptyset. \end{aligned}$$

(B3) [Rank of the diagonal blocks]

$$\begin{aligned} \mathrm{rank}\, \bar{A}[R_0, C_0] &= |R_0|, \\ \mathrm{rank}\, \bar{A}[R_k, C_k] &= |R_k| = |C_k| \quad \text{for} \quad k = 1, \ldots, b, \\ \mathrm{rank}\, \bar{A}[R_\infty, C_\infty] &= |C_\infty|. \end{aligned}$$

(B4) [Uniqueness] $\bar{A}$ *is the finest block-triangular matrix with properties (B2) and (B3) that is LM-equivalent to* A. *Namely, if* $\hat{A}$ *is LM-equivalent to* A *which is block-triangularized with respect to certain partitions*

$$(\hat{R}_0; \hat{R}_1, \ldots, \hat{R}_q; \hat{R}_\infty), \quad (\hat{C}_0; \hat{C}_1, \ldots, \hat{C}_q; \hat{C}_\infty)$$

of $\mathrm{Row}(\hat{A})$ *and* $\mathrm{Col}(\hat{A})$ $(= \mathrm{Col}(A))$ *with the diagonal blocks satisfying the conditions (B2) and (B3), then* $\hat{C}_k$ *is a union of the blocks defined by* $L(p_R)$. □

The matrix $\bar{A}$ above is the CCF of A. The CCF is uniquely determined so far as the partitions of the row and column sets as well as the partial order among the blocks are concerned, whereas there remains some indeterminacy, or degree of freedom, in the numerical values of the entries in the Q-part (for example, elementary row transformations within a block change numerical values without affecting the block structure). See $\bar{A}_1$ and $\bar{A}_2$ in Example 5 below. When the numerical indeterminacy is to be emphasized, such $\bar{A}$ will be called *a* CCF, instead of *the* CCF. We make use of such indeterminacy in Theorem 7.

The submatrices $\bar{A}[R_0, C_0]$ and $\bar{A}[R_\infty, C_\infty]$ are called the *horizontal tail* and the *vertical tail*, respectively. The tails are nonsquare if they are not empty, and (B1) and (B3) imply that

$$\mathrm{rank}\, A = \mathrm{rank}\, \bar{A} = |C| - \delta_0 = |R| - \delta_\infty$$

with

$$\delta_0 = |C_0| - |R_0|, \quad \delta_\infty = |R_\infty| - |C_\infty|.$$

Hence A is nonsingular if and only if $C_0 = R_\infty = \emptyset$. In Example 3 we have $C_0 = R_\infty = \emptyset$, and the number of square blocks $b = 5$.

Example 4 Consider a 4×5 LM-matrix $A = \binom{Q}{T}$ with

$$Q = \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 2 & 1 & 1 & 0 \end{pmatrix} \end{array}, \quad T = \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{pmatrix} t_1 & 0 & 0 & 0 & t_2 \\ 0 & t_3 & 0 & 0 & t_4 \end{pmatrix} \end{array}.$$

By choosing $S = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ in the admissible transformation (4), we obtain the CCF:

$$\bar{A} = \begin{array}{c} \begin{array}{ccccc} x_3 & x_4 & x_1 & x_2 & x_5 \end{array} \\ \begin{pmatrix} 1 & 1 & & 2 & \\ & & 1 & -1 & 0 \\ & & t_1 & 0 & t_2 \\ & & 0 & t_3 & t_4 \end{pmatrix} \end{array}$$

with a nonempty horizontal tail $C_0 = \{x_3, x_4\}$, a single ($b = 1$) square block $C_1 = \{x_1, x_2, x_5\}$, and an empty vertical tail $C_\infty = \emptyset$. It is not difficult to verify that $L(p_R) = \{C_0, C_0 \cup C_1\}$. Example 7 in §6 will illustrate how the CCF, as well as the matrix S, can be found efficiently. ☐

Here we mention an extension of the notion of LM-matrix and its CCF when the base field is replaced by a ring. Let $\boldsymbol{D}$ be an integral domain [52], and $\boldsymbol{K}$ the field of quotients of $\boldsymbol{D}$; it is still assumed that $\boldsymbol{K}$ is a subfield of $\boldsymbol{F}$. We say that a matrix $A = \binom{Q}{T}$ is an LM-matrix with respect to $\boldsymbol{F}/\boldsymbol{D}$, denoted as $A \in \mathrm{LM}(\boldsymbol{F}/\boldsymbol{D})$, if $A \in \mathrm{LM}(\boldsymbol{F}/\boldsymbol{K})$ and furthermore, Q is a matrix over $\boldsymbol{D}$. Accordingly the admissible transformation over $\boldsymbol{D}$ is defined to be a transformation of the form (4) with S being a matrix over $\boldsymbol{D}$ with $\det S \neq 0$. Then the matrix resulting from this transformation is again an LM-matrix with respect to $\boldsymbol{F}/\boldsymbol{D}$. Note, however, that an admissible transformation over $\boldsymbol{D}$ is not always invertible since the inverse of S may not exist among the matrices over $\boldsymbol{D}$. The matrix S has its inverse S^{-1} over $\boldsymbol{D}$ if and only if $\det S$ is an invertible element of $\boldsymbol{D}$, in which case S is called *unimodular* over $\boldsymbol{D}$.

It follows easily from Theorem 6 (see also Example 5 below) that for $A \in \mathrm{LM}(\boldsymbol{F}/\boldsymbol{D})$ there exists an admissible transformation over $\boldsymbol{D}$, which is not necessarily invertible, such that the resulting matrix $\bar{A}$ agrees with a CCF of A as an LM-matrix with respect to $\boldsymbol{F}/\boldsymbol{K}$.

When the invertibility is imposed upon the admissible transformation, we can still claim a similar statement when $\boldsymbol{D}$ is a principal ideal domain (PID) [52]; the ring of integers $\mathbf{Z}$ and the ring of univariate polynomials over a field are typical examples of a PID. It should be clear that a linear extension of a partial order means a linear order (=total order) that is compatible with the partial order, also called a topological sorting in computer science. Our indexing convention (20) for the blocks $\{C_k\}$ in the CCF of A represents a linear extension of the partial order $\preceq$ in the CCF. The following fact was observed by Murota [36] (see also [39]) in the case where $\boldsymbol{D}$ is a ring of polynomials. The proof will be given later in the next subsection.

Theorem 7 *Let A be an LM-matrix with respect to $\boldsymbol{F}/\boldsymbol{D}$, where $\boldsymbol{D}$ is a PID. Let $\{C_k\}_{k=0}^{\infty}$ denote the partition of C in the CCF of A and $\preceq$ the partial order among the blocks (using the notation of Theorem 6). For any linear extension of $\preceq$, which is represented by the linear order of the index k of the blocks, there exist permutation matrices $P_{\rm r}$ and $P_{\rm c}$, a unimodular matrix S over $\boldsymbol{D}$, and a CCF $\bar{A}$ of A (as an LM-matrix with respect to $\boldsymbol{F}/\boldsymbol{K}$) such that*

$$\hat{A} = P_{\rm r} \begin{pmatrix} S & O \\ O & I \end{pmatrix} \begin{pmatrix} Q \\ T \end{pmatrix} P_{\rm c}$$

is in the same block-triangular form as $\bar{A}$, having the same diagonal blocks, i.e., $\hat{A}[R_k, C_l] = \bar{A}[R_k, C_l] = O$ for $k > l$ and $\hat{A}[R_k, C_k] = \bar{A}[R_k, C_k]$ for $k = 0, 1, \ldots, b, \infty$. (It is not claimed that $\hat{A}[R_k, C_l]$ coincides with $\bar{A}[R_k, C_l]$ for $k < l$.) □

Example 5 Let $\boldsymbol{D} = \mathbf{Z}$, $\boldsymbol{K} = \mathbf{Q}$ and $\boldsymbol{F} = \mathbf{Q}(t_1, t_2)$, where t_1 and t_2 are indeterminates. Consider a 3×3 LM-matrix with respect to $\boldsymbol{F}/\mathbf{Z}$:

$$A = \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{pmatrix} 2 & -2 & -4 \\ 3 & 1 & 2 \\ 0 & t_1 & t_2 \end{pmatrix} \end{array}.$$

First regard A as a member of LM($\boldsymbol{F}/\mathbf{Q}$). By choosing $S = S_1 = \begin{pmatrix} 1/4 & 1/2 \\ -3/2 & 1 \end{pmatrix}$ (with $\det S_1 = 1$) in the admissible transformation (4) we obtain a CCF:

$$\bar{A}_1 = \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{pmatrix} 2 & & \\ & 4 & 8 \\ & t_1 & t_2 \end{pmatrix} \end{array},$$

which has two square blocks $C_1 = \{x_1\}$ and $C_2 = \{x_2, x_3\}$ with no order relation between them.

The transformation using $S = S_1$ is not admissible over $\mathbf{Z}$. However, an admissible transformation over $\mathbf{Z}$ can be constructed easily by putting $S = S_2 = 4 \cdot S_1$, which yields another CCF:

$$\bar{A}_2 = \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{pmatrix} 8 & & \\ & 16 & 32 \\ & t_1 & t_2 \end{pmatrix} \end{array}.$$

It is noted however that the admissible transformation with $S = S_2$ is not invertible since S_2 is not unimodular with $\det S_2 = 16$.

Restricting S to a unimodular matrix over $\mathbf{Z}$, we may take $S = S_3 = \begin{pmatrix} -1 & 1 \\ -3 & 2 \end{pmatrix}$ (with $\det S_3 = 1$) to transform A to a block-triangular matrix

$$\hat{A} = \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{pmatrix} 1 & 3 & 6 \\ & 8 & 16 \\ & t_1 & t_2 \end{pmatrix} \end{array}$$

with order relation $C_1 = \{x_1\} \preceq C_2 = \{x_2, x_3\}$. This matrix $\hat{A}$ has the same diagonal blocks with

$$\bar{A}_3 = \begin{array}{c} \\ \\ \\ \end{array}\!\!\begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{pmatrix} 1 & & \\ & 8 & 16 \\ & t_1 & t_2 \end{pmatrix} \end{array},$$

which is another CCF of A obtained with $S = S_3 = \begin{pmatrix} 1/8 & 1/4 \\ -3 & 2 \end{pmatrix}$. $\square$

3.2 Construction of CCF

This subsection gives a sketch of the constructive proof of Theorem 6. A complete proof can be found in [26], [28], [41]. It should be emphasized that the following mathematical construction of the CCF can be polished up to a practically efficient algorithm, which will be described in §6.

First note that the admissible transformation (4) for $A \in \mathrm{LM}(\boldsymbol{F}/\boldsymbol{K}; m_Q, m_T, n)$ is equivalent to

$$P_{\mathrm{r}} \begin{pmatrix} S & O \\ O & P_T \end{pmatrix} \begin{pmatrix} Q \\ T \end{pmatrix} P_{\mathrm{c}},$$

which contains another permutation matrix P_T. In what follows we will find these four matrices $P_{\mathrm{c}}, S, P_T, P_{\mathrm{r}}$ that bring about the CCF.

[Matrix P_{c}]: As has been explained in §3.1, the submodular function p_R determines a sublattice $L(p_R)$, which in turn yields a pair

$$\mathcal{P}(L(p_R)) = (\{C_0; C_1, \ldots, C_b; C_\infty\}, \preceq) \tag{21}$$

of a partition of $C = \mathrm{Col}(A) = \mathrm{Col}(Q) = \mathrm{Col}(T)$ and a partial order (see (19)). Recall the relation (18): $X_k = \cup_{l=0}^{k} C_l$ $(0 \leq k \leq b)$ as well as (20). The permutation matrix P_{c} is such that the column set C is reordered as $C_0, C_1, \ldots, C_b, C_\infty$, where the ordering within each block is arbitrary.

[Matrix S]: We use a short-hand notation $\rho(J) = \rho(R_Q, J) = \mathrm{rank}\, Q[R_Q, J]$ for $J \subseteq C$. Put $Q_0 = QP_{\mathrm{c}}$, where $\mathrm{Col}(Q_0)$ is identified with C through permutation P_{c}. Since $Q_0[\mathrm{Row}(Q_0), C_0]$ contains $\rho(X_0)$ independent row vectors and the others are linearly dependent on them, we can find a nonsingular matrix $S_0 \in \mathrm{GL}(m_Q, \boldsymbol{K})$ such that $Q_1 = S_0 Q_0$ satisfies

$$Q_1[\mathrm{Row}(Q_1) - R_{Q0}, C_0] = O,$$

$$\mathrm{rank}\, Q_1[R_{Q0}, C_0] = |R_{Q0}| = \rho(X_0)$$

for some $R_{Q0} \subseteq \mathrm{Row}(Q_1)$; that is,

$$Q_1 = S_0 Q_0 = \begin{array}{c|cc} & C_0 & \overline{C_0} \\ \hline R_{Q0} & [\equiv] & * \\ \overline{R_{Q0}} & O & * \end{array},$$

where $\overline{R_{Q0}} = \mathrm{Row}(Q_1) - R_{Q0}$, $\overline{C_0} = C - C_0$ and $[\equiv]$ indicates a submatrix with independent rows.

Next, since $Q_1[\mathrm{Row}(Q_1), C_0 \cup C_1]$ contains $\rho(X_1)$ independent row vectors, we can find $S_1 \in \mathrm{GL}(m_Q, \boldsymbol{K})$ such that $Q_2 = S_1 Q_1$ satisfies

$$Q_2[\mathrm{Row}(Q_2) - R_{Q0}, C_0] = O,$$

$$Q_2[\mathrm{Row}(Q_2) - (R_{Q0} \cup R_{Q1}), C_0 \cup C_1] = O;$$

$$\begin{aligned} \mathrm{rank}\, Q_2[R_{Q0}, C_0] &= |R_{Q0}| = \rho(X_0), \\ \mathrm{rank}\, Q_2[R_{Q1}, C_1] &= |R_{Q1}| = \rho(X_1) - \rho(X_0) \end{aligned}$$

for some $R_{Q0}, R_{Q1} \subseteq \mathrm{Row}(Q_2)$ with $R_{Q0} \cap R_{Q1} = \emptyset$. That is,

$$Q_2 = S_1 Q_1 = \begin{array}{r|ccc} & C_0 & C_1 & \overline{C_0 \cup C_1} \\ \hline R_{Q0} & [\equiv] & \triangle & * \\ R_{Q1} & O & [\equiv] & * \\ \overline{R_{Q0} \cup R_{Q1}} & O & O & * \end{array}.$$

We may further impose that

$$\begin{array}{l} \text{The nonzero row vectors of } Q_2[R_{Q0}, C_1] \text{ (indicated by } \triangle \text{ above)} \\ \text{are linearly independent of the row vectors of } Q_2[R_{Q1}, C_1], \end{array} \tag{22}$$

for otherwise we could eliminate the former with the latter.

Continuing such sweep-out operations, we can find a nonsingular matrix $S \in \mathrm{GL}(m_Q, \boldsymbol{K})$ and a partition of $\mathrm{Row}(\bar{Q})$:

$$(R_{Q0}; R_{Q1}, \ldots, R_{Qb}; R_{Q\infty}) \tag{23}$$

such that $\bar{Q} = SQP_{\mathrm{c}}$ satisfies

$$\bar{Q}[R_{Ql}, C_k] = O \quad (0 \le k < l \le \infty); \tag{24}$$

$$\begin{aligned} \mathrm{rank}\, \bar{Q}[R_{Q0}, C_0] &= |R_{Q0}| = \rho(X_0), \\ \mathrm{rank}\, \bar{Q}[R_{Qk}, C_k] &= |R_{Qk}| = \rho(X_k) - \rho(X_{k-1}) \quad (k = 1, \ldots, b), \\ |R_{Q\infty}| &= m_Q - \rho(X_b). \end{aligned} \tag{25}$$

We may further impose that

$$\begin{array}{l} \text{For } 0 \le k < l \le \infty \text{, the nonzero row vectors of } \bar{Q}[R_{Qk}, C_l] \text{ are} \\ \text{linearly independent of the row vectors of } \bar{Q}[R_{Ql}, C_l]. \end{array} \tag{26}$$

[Matrix P_T]: Define a partition of R_T:

$$(R_{T0}; R_{T1}, \ldots, R_{Tb}; R_{T\infty}) \tag{27}$$

by

$$\begin{aligned} R_{T0} &= \Gamma(R_T, X_0), \\ R_{Tk} &= \Gamma(R_T, X_k) - \Gamma(R_T, X_{k-1}) \quad (k = 1, \ldots, b), \\ R_{T\infty} &= \mathrm{Row}(T) - \Gamma(R_T, X_b) \end{aligned} \tag{28}$$

using the Γ of (6). Let P_T be a permutation matrix which permutes $\mathrm{Row}(T)$ compatibly with (27). Then $\bar{T} = P_T T P_{\mathrm{c}}$ is in an explicit block-triangular form:

$$T[R_{Tl}, C_k] = \bar{T}[R_{Tl}, C_k] = O \quad (0 \le k < l \le \infty), \tag{29}$$

where we identify $\mathrm{Row}(\bar{T}) = \mathrm{Row}(T)$ and $\mathrm{Col}(\bar{T}) = \mathrm{Col}(T) = C$.

[Matrix P_{r}]: So far we have constructed two block-triangular matrices $\bar{Q}$ and $\bar{T}$, the former being block-triangularized with respect to the partitions (21) and (23) and the latter with respect to (21) and (27). Put these two matrices together:

$$\bar{A} = \begin{pmatrix} \bar{Q} \\ \bar{T} \end{pmatrix},$$

and consider a partition of $\mathrm{Row}(\bar{A})$:

$$(R_0; R_1, \ldots, R_b; R_\infty), \tag{30}$$

where $R_k = R_{Qk} \cup R_{Tk}$ for $k = 0, 1, \ldots, b, \infty$. By (24) and (29), $\bar{A}$ is (essentially) block-triangularized with respect to the partitions (21) and (30), namely,

$$\bar{A}[R_l, C_k] = O \quad (0 \le k < l \le \infty).$$

The matrix P_{r} is to rearrange $\mathrm{Row}(\bar{A})$ compatibly with (30).

The block-triangular matrix $\bar{A}$ constructed in this way is obviously LM-equivalent to A. Based on the rank formula of Theorem 5 we can show that this matrix enjoys the properties (B2) to (B4). We will indicate the essense here, referring the reader to [28], pp. 177–179, for the complete proof. In addition to $\rho(J)$ we use another short-hand notation $\gamma(J) = \gamma(R_T, J)$ for $J \subseteq C$.

Consider the horizontal tail $\bar{A}[R_0, C_0]$. Since $C_0 \in L(p_R)$, $\rho(C_0) = |R_{Q0}|$ and $\gamma(C_0) = |R_{T0}|$, we have

$$0 = p_R(\emptyset) \ge \min p_R = p_R(C_0) = \rho(C_0) + \gamma(C_0) - |C_0| = |R_0| - |C_0|,$$

which, combined with Theorem 5, implies

$$\mathrm{rank}\, \bar{A}[R_0, C_0] = \mathrm{rank}\, \bar{A}[\mathrm{Row}(\bar{A}), C_0] = \mathrm{rank}\, A[R, C_0] = \min p_R + |C_0| = |R_0|.$$

This shows in particular that $|R_0| \le |C_0|$. If the equality holds here, then $p_R(\emptyset) = \min p_R$, i.e., $\emptyset \in L(p_R)$. Since $C_0 = \min L(p_R)$, this implies $C_0 = \emptyset$ and therefore $R_0 = \emptyset$. Hence follow (B2) and (B3) for the horizontal tail.

For the first square block $\bar{A}[R_1, C_1]$ we note that $p_R(C_0) = p_R(C_0 \cup C_1) = \min p_R$. This shows

$$\begin{aligned} |R_1| &= |R_0 \cup R_1| - |R_0| \\ &= (\rho(C_0 \cup C_1) - \rho(C_0)) + (\gamma(C_0 \cup C_1) - \gamma(C_0)) \\ &= p_R(C_0 \cup C_1) - p_R(C_0) + |C_1| = |C_1|. \end{aligned}$$

It also follows from Theorem 5, as well as the relation: $\min p_R = |R_0| - |C_0|$ shown above, that

$$\begin{aligned} \mathrm{rank}\, \bar{A}[R_0 \cup R_1, C_0 \cup C_1] &= \mathrm{rank}\, \bar{A}[\mathrm{Row}(\bar{A}), C_0 \cup C_1] \\ &= \mathrm{rank}\, A[R, C_0 \cup C_1] \\ &= \min p_R + |C_0| + |C_1| = |R_0| + |R_1|. \end{aligned}$$

This shows $\operatorname{rank} \bar{A}[R_1, C_1] = |R_1|$ since $\operatorname{rank} \bar{A}[R_0, C_0] = |R_0|$.

The conditions (B2) and (B3) for the remaining blocks can be shown similarly. The invariance of p_R explained after (13) is the key to prove the uniqueness (B4); see [28], p. 179. We may mention that the argument above conforms with the Jordan-Hölder type decomposition principle for submodular functions developed by Iri [19], Nakamura [43], Tomizawa [51].

A similar argument establishes Theorem 7, which is concerned with the block-triangularization with respect to a unimodular transformation over a PID. The Hermite normal form [44], [50] under a unimodular transformation guarantees the existence of a unimodular matrix S such that $\bar{Q} = SQP_{\mathrm{c}}$ satisfies (24) and (25). However, we cannot impose the further condition (22) or (26), which fact causes the discrepancy in the upper off-diagonal blocks of $\hat{A}$ and $\bar{A}$.

4 Irreducibility

In this section we investigate into the notion of LM-irreducibility. Most of the results below are natural extensions of the results concerning the full indecomposability (or DM-irreducibility) of a generic matrix. See Schneider [49] for a historical account on the notion of full indecomposability.

First recall the definition (see §1) of the LM-irreducibility with respect to the admissible transformation. Namely, an LM-matrix A is LM-irreducible (or simply irreducible) if it does not split into more than one nonempty block under the admissible transformation, or more precisely, if any LM-matrix A' that is LM-equivalent to A is fully indecomposable.

With reference to the CCF, $\bar{A}$, of A, we see that each block $\bar{A}[R_k, C_k]$ of the CCF is irreducible ($k = 0, 1, \ldots, b, \infty$) using the notation of Theorem 6. Hence, A is irreducible if (a) $b = 1$ and $C_0 = R_\infty = \emptyset$, (b) $b = 0$ and $R_\infty = \emptyset$, or (c) $b = 0$ and $C_0 = \emptyset$.

Combining this observation with Theorem 6(B1) we obtain the following characterization of LM-irreducibility in terms of the lattice $L(p_R)$ of minimizers of p_R. This is a kind of "dual" characterization of the LM-irreducibility as opposed to the "primal" characterization (definition) in terms of the indecomposability with respect to the admissible transformation.

Theorem 8 *Let* $A \in \mathrm{LM}(\boldsymbol{F}/\boldsymbol{K})$.
(a) *In case* $|R| = |C|$*: A is LM-irreducible* $\iff$ $L(p_R) = \{\emptyset, C\}$*;*
(b) *In case* $|R| < |C|$*: A is LM-irreducible* $\iff$ $L(p_R) = \{C\}$*;*
(c) *In case* $|R| > |C|$*: A is LM-irreducible* $\iff$ $L(p_R) = \{\emptyset\}$. □

This characterization will be rephrased in a more algorithmic statement later in Theorem 17.

The following theorem refers to the rank of submatrices of an LM-irreducible matrix. This is an extension of the result due to Marcus-Minc [22] and to Brualdi [4] for a generic matrix (cf. p.112 of [5]); see also Theorem 4.2.2 of [5].

Theorem 9 *Let* $A \in \mathrm{LM}(\boldsymbol{F}/\boldsymbol{K})$ *be LM-irreducible.*
(a) *In case* $|R| = |C|$*:* $\mathrm{rank}\, A[R - \{i\}, C - \{j\}] = |R| - 1 \quad (\forall i \in R, \forall j \in C)$*;*
(b) *In case* $|R| < |C|$*:* $\mathrm{rank}\, A[R, C - \{j\}] = |R| \quad (\forall j \in C)$*;*
(c) *In case* $|R| > |C|$*:* $\mathrm{rank}\, A[R - \{i\}, C] = |C| \quad (\forall i \in R)$.

(Proof) (a) Put $R' = R - \{i\}$, $C' = C - \{j\}$ and suppose that $A[R', C']$ were singular. Then, by Theorem 5, $p(R', J') \leq -1$ for some J' ($\emptyset \neq J' \subseteq C'$). On the other hand, it follows from

$$p(R, J') - p(R', J') = \begin{cases} \rho(R_Q, J') - \rho(R_Q - \{i\}, J') & (\text{if } i \in R_Q) \\ \gamma(R_T, J') - \gamma(R_T - \{i\}, J') & (\text{if } i \in R_T) \end{cases}$$

that $p(R, J') - p(R', J') \leq 1$. Hence $p(R, J') \leq p(R', J') + 1 \leq 0$, which would imply $J' \in L(p_R)$, a contradition to Theorem 8(a). The proofs for (b), (c) are similar; see [29]. □

As immediate corollaries we obtain the following properties of a nonsingular irreducible LM-matrix. We regard the determinant of $A \in \mathrm{LM}(\boldsymbol{F}/\boldsymbol{K})$ as a polynomial in $\mathcal{T}$ (=set of nonzero entries of T) with coefficients from the base field $\boldsymbol{K}$.

Theorem 10 *Let* $A \in \mathrm{LM}(\boldsymbol{F}/\boldsymbol{K})$ *be nonsingular and LM-irreducible.*
(1) A^{-1} *is completely dense, i.e.,* $(A^{-1})_{ji} \neq 0,\ \forall\,(i,j)$.
(2) *Each element of* $\mathcal{T}$ *appears in* $\det A$. □

The following theorem of Murota [29] states to the effect that the combinatorial irreducibility (namely LM-irreducibility) is essentially equivalent to the algebraic irreducibility of the determinant. This is an extension of the result of Frobenius [12] for a generic matrix (see also [5], [47], [48], [49]).

Theorem 11 *Let* $A \in \mathrm{LM}(\boldsymbol{F}/\boldsymbol{K})$ *be nonsingular. The determinant* $\det A$ *is an irreducible polynomial in the ring* $\boldsymbol{K}[\mathcal{T}]$ *if* A *is LM-irreducible. Conversely, if* $\det A$ *is an irreducible polynomial, then there exists in the CCF of* A *at most one diagonal block which contains elements of* $\mathcal{T}$ *and all the other diagonal blocks are* 1×1 *matrices over* $\boldsymbol{K}$.

(Proof) The proof for the first half is long; see [29]. The second half follows easily from Theorem 6 and Theorem 10(2). □

A minor (=subdeterminant) of $A \in \mathrm{LM}(\boldsymbol{F}/\boldsymbol{K})$ is also a polynomial in $\mathcal{T}$ over $\boldsymbol{K}$. Let $d_k(\mathcal{T}) \in \boldsymbol{K}[\mathcal{T}]$ denote the k-th determinantal divisor of A, i.e., the greatest common divisor of all minors of order k in A as polynomials in $\mathcal{T}$ over $\boldsymbol{K}$. Note that $d_k(\mathcal{T}) \in \boldsymbol{K}^* = \boldsymbol{K} - \{0\}$ means $d_k(\mathcal{T})$ is a "constant" free from any variables in $\mathcal{T}$.

Theorem 12 *Let* $A \in \mathrm{LM}(\boldsymbol{F}/\boldsymbol{K})$ *be LM-irreducible.*
(a) *In case* $|R| = |C|$: $d_k(\mathcal{T}) \in \boldsymbol{K}^*$ *for* $k = 1, \ldots, |R| - 1$;
(b) *In case* $|R| < |C|$: $d_k(\mathcal{T}) \in \boldsymbol{K}^*$ *for* $k = 1, \ldots, |R|$;
(c) *In case* $|R| > |C|$: $d_k(\mathcal{T}) \in \boldsymbol{K}^*$ *for* $k = 1, \ldots, |C|$.

(Proof) (a) It suffices to show that $d_k(\mathcal{T})$ is free from any $t \in \mathcal{T}$ for $k = |R| - 1$. Suppose t appears at position (i,j). It follows from Theorem 9(a) that $\delta \equiv \det A[R - \{i\}, C - \{j\}] \neq 0$. Obviously δ does not contain t, and, a fortiori, $d_k(\mathcal{T})$ does not contain t, since $d_k(\mathcal{T})$ is a divisor of δ. (b) and (c) can be proven similarly using Theorem 9. □

For a general (reducible) LM-matrix Theorems 6, 11 and 12 together imply the following.

Theorem 13 *Let* r *be the rank of* $A \in \mathrm{LM}(\boldsymbol{F}/\boldsymbol{K})$. *Then* $d_k(\mathcal{T}) \in \boldsymbol{K}^*$ *for* $k = 1, \ldots, r-1$, *and the decomposition of the* r*-th determinantal divisor* $d_r(\mathcal{T})$ *of* A *into irreducible factors in the ring* $\boldsymbol{K}[\mathcal{T}]$ *is given by*

$$d_r(\mathcal{T}) = \alpha \cdot \prod_{l=1}^{b} \det \bar{A}[R_l, C_l],$$

where $\bar{A}[R_l, C_l]$ $(l = 1, \ldots, b)$ *are the irreducible square blocks in the CCF of* A, *and* $\alpha \in \boldsymbol{K}^*$. *(Exactly speaking, those factors on the right-hand side which belong to* $\boldsymbol{K}$ *should not be counted as irreducible factors in* $\boldsymbol{K}[\mathcal{T}]$ *since they are invertible elements in* $\boldsymbol{K}[\mathcal{T}]$.*)* □

5 Further Properties

5.1 Principal structure of LM-matrices

A submatrix $A[I, C]$ (with $I \subseteq R$) of an LM-matrix A is again an LM-matrix, for which the CCF is defined. The CCF of $A[I, C]$, in turn, defines a partition of C, which varies with I. Let us denote by $\mathcal{P}_{\rm CCF}(I)$ (cf. (21)) the pair of the partition of C and the partial order among the blocks in the CCF of the submatrix $A[I, C]$. Here we are interested in the family $\{\mathcal{P}_{\rm CCF}(I) \mid I \in \mathcal{B}\}$ of partitions, where

$$\mathcal{B} = \{I \subseteq R \mid \operatorname{rank} A = \operatorname{rank} A[I, C] = |I|\},$$

which denotes the family of row-bases of A. The theorem below gives a concise characterization of the coarsest common refinement of $\{\mathcal{P}_{\rm CCF}(I) \mid I \in \mathcal{B}\}$ in terms of the submodular function p_R associated with the whole matrix A.

The characterization refers to the notion of "principal structure of a submodular system" introduced by Fujishige [14], [15]. For $j \in C$ consider the family L_j of the minimizers of p_R over $\{J \subseteq C \mid J \ni j\}$:

$$L_j = \{J \subseteq C \mid J \ni j; \quad p_R(J) \leq p_R(J') \quad \forall J' \ni j\},$$

which forms a sublattice of 2^C because of the submodularity (13) of p_R. Denote by $D(j)$ the (uniquely determined) smallest set of L_j. The binary relation $\preceq_{p_R}$ on C defined by $[i \preceq_{p_R} j \iff i \in D(j)]$, or equivalently by $[i \preceq_{p_R} j \iff D(i) \subseteq D(j)]$, is a quasi-order, being reflexive and transitive. Then the equivalence relation defined by $[i \preceq_{p_R} j$ and $j \preceq_{p_R} i]$ determines a partition of C into blocks, among which a partial order is induced from the original quasi-order. This is called the *principal structure*, to be denoted as $\mathcal{P}_{\rm PS}$, of the submodular system of (C, p_R).

The following theorem of Murota [32] shows that the coarsest common refinement of $\{\mathcal{P}_{\rm CCF}(I) \mid I \in \mathcal{B}\}$ agrees with the principal structure of the submodular system of (C, p_R).

Theorem 14

$$\mathcal{P}_{\rm PS} = \bigwedge_{I \in \mathcal{B}} \mathcal{P}_{\rm CCF}(I),$$

where the right-hand side designates the coarsest partition of C which is finer than all $\mathcal{P}_{\rm CCF}(I)$ with $I \in \mathcal{B}$. □

In view of the correspondence (as explained in §3.1) between the family of partitions $\{\mathcal{P}_{\rm CCF}(I) \mid I \in \mathcal{B}\}$ and the family of sublattices $\{L(p_I) \mid I \in \mathcal{B}\}$, we can think of this theorem as a characterization of the sublattice generated by $\{L(p_I) \mid I \in \mathcal{B}\}$.

The essential content of the above theorem for the special case of a generic matrix $A = T$ (with $m_Q = 0$) has been obtained by McCormick [23] (without reference to the notion of principal structure).

Example 6 Consider a 5×3 LM-matrix (with base field $\mathbf{Q}$):

$$A = \begin{array}{c} \\ r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{array} \!\!\! \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left(\begin{array}{ccc} 1 & 2 & 1 \\ 1 & 1 & -1 \\ 0 & t_1 & t_2 \\ 0 & t_3 & t_4 \\ t_5 & t_6 & 0 \end{array} \right) \end{array},$$

where $C = \{x_1, x_2, x_3\}$, $R = \{r_1, r_2, r_3, r_4, r_5\}$, and t_i $(i = 1, \ldots, 6)$ are indeterminates. This matrix is LM-irreducible, the whole matrix being a vertical tail.

For a nonsingular submatrix $A[I, C]$ with $I = \{r_1, r_2, r_3\}$, we obtain its CCF

$$\begin{array}{c|ccc} & x_1 & x_2 & x_3 \\ \hline r_1 & 1 & 2 & 1 \\ r_2 & & -1 & -2 \\ r_3 & & t_1 & t_2 \end{array}$$

by subtracting row r_1 from row r_2 in $A[I, C]$. Hence, $\mathcal{P}_{\mathrm{CCF}}(I)$ is given by $\{x_1\} \prec \{x_2, x_3\}$.

By inspection we see that $\mathcal{B} = \{I \subset R \mid |I| = 3\}$. $\mathcal{P}_{\mathrm{CCF}}(I)$ for all $I \in \mathcal{B}$ are given as follows.

$I \in \mathcal{B}$	$\mathcal{P}_{\mathrm{CCF}}(I)$
$\{r_1, r_2, r_5\}$	$\{x_3\} \prec \{x_1, x_2\}$
$\{r_i, r_j, r_5\} (i = 1, 2; j = 3, 4)$	$\{x_1, x_2, x_3\}$
Otherwise	$\{x_1\} \prec \{x_2, x_3\}$

This shows that $\bigwedge_{I \in \mathcal{B}} \mathcal{P}_{\mathrm{CCF}}(I)$ is given by $\{x_1\} \prec \{x_2\}$, $\{x_3\} \prec \{x_2\}$. On the other hand, we have $D(x_1) = \{x_1\}$, $D(x_2) = C$, $D(x_3) = \{x_3\}$ since $p_R(\emptyset) = 0$, $p_R(\{x_1\}) = 1$, $p_R(\{x_2\}) = 3$, $p_R(\{x_3\}) = 2$, $p_R(\{x_1, x_2\}) = 3$, $p_R(\{x_1, x_3\}) = 3$, $p_R(\{x_2, x_3\}) = 3$, $p_R(C) = 2$. Hence $\mathcal{P}_{\mathrm{PS}}$ agrees with $\{x_1\} \prec \{x_2\}$, $\{x_3\} \prec \{x_2\}$. Note also that $\mathcal{P}_{\mathrm{PS}} \neq \mathcal{P}_{\mathrm{CCF}}(I)$ for each $I \in \mathcal{B}$. □

5.2 Properties of mixed matrices

In this subsection $A = Q + T$ denotes a mixed matrix with respect to $\boldsymbol{F}/\boldsymbol{K}$, with $\mathrm{Row}(A) = R$ and $\mathrm{Col}(A) = C$.

If A is nonsingular, it can be decomposed into LU-factors as $P_{\mathrm{r}}\ A\ P_{\mathrm{c}} = L\ U$ with suitable permutation matrices P_{r} and P_{c}. In general the entries of the matrices L and U are rational functions in $\mathcal{T}$ (=set of nonzero entries of T) over $\boldsymbol{K}$. If all the diagonal entries of L and U belong to $\boldsymbol{K}$, then obviously $\det A \in \boldsymbol{K}$. The following theorem of Murota [24] asserts that the converse is also true (see [24], [28] for the proof). Recall the notation $\boldsymbol{K}^* = \boldsymbol{K} - \{0\}$.

Theorem 15 *Let $A = Q + T$ be a mixed matrix with base field $\boldsymbol{K}$. Then $\det A \in \boldsymbol{K}^*$ if and only if there exist permutation matrices P_{r} and P_{c}, and LU-factors L and U: $P_{\mathrm{r}}\ A\ P_{\mathrm{c}} = L\ U$ such that (i) $L_{ii} = 1$ and $L_{ij} = 0$ for $i < j$; (ii) U_{ij} is a polynomial (of degree at most one) in $\mathcal{T}$ over $\boldsymbol{K}$ for $i > j$, $U_{ii} \in \boldsymbol{K}^*$, and $U_{ij} = 0$ for $i > j$.* □

The final theorem of this section is an extension of the "determinantal version of the Frobenius-König theorem" due to Hartfiel-Loewy [17], who established it in the case where A is a square mixed matrix. Their original proof (for square case) is quite involved based on factorizations of determinants. Here we provide an alternative proof using the rank formula of Theorem 5 for LM-matrices.

Theorem 16 *Let* $A = Q + T$ *be a mixed matrix. Then* $\mathrm{rank}\, A < |C|$ *if and only if* (i) $T[I, J] = O$ *and* (ii) $\mathrm{rank}\, Q[I, J] < |I| + |J| - |R|$ *for some* $I \subseteq R$ *and* $J \subseteq C$*. Similarly,* $\mathrm{rank}\, A < |R|$ *if and only if* (i) $T[I, J] = O$ *and* (ii) $\mathrm{rank}\, Q[I, J] < |I| + |J| - |C|$ *for some* $I \subseteq R$ *and* $J \subseteq C$*.*

(Proof) Consider the LM-matrix $\tilde{A}$ of (3) for A and let $p : 2^{\tilde{R}} \times 2^{R \cup C} \to \mathbf{Z}$ be the function defined for $\tilde{A}$ as in (11), where $\tilde{R} = \mathrm{Row}(\tilde{A})$ and we identify $\mathrm{Col}(\tilde{A})$ with $R \cup C$. Then

$$p(\tilde{R}, (R - I) \cup J) = \mathrm{rank}\, Q[I, J] + |(R - I) \cup \Gamma(R, J)| - |J|$$

with $\Gamma(R, J) = \{i \in R \mid \exists j \in J : T_{ij} \neq 0\}$ (cf. (6)), where it should be clear that $I \subseteq \mathrm{Col}(\tilde{A})$ on the left-hand side and $I \subseteq \mathrm{Row}(A)$ on the right-hand side. Theorem 5 applied to $\tilde{A}$ implies that $\mathrm{rank}\, \tilde{A} < |R| + |C|$ if and only if $p(\tilde{R}, (R - I) \cup J) < 0$ for some $I \subseteq R$ and $J \subseteq C$. In the latter condition we may assume that $I \cap \Gamma(R, J) = \emptyset$, i.e., (i) $T[I, J] = O$, and then $p(\tilde{R}, (R - I) \cup J) < 0$ reduces to (ii) $\mathrm{rank}\, Q[I, J] < |I| + |J| - |R|$. The proof for the first claim is completed by the obvious relation: $\mathrm{rank}\, \tilde{A} = \mathrm{rank}\, A + |R|$. The second claim follows from the first applied to A transposed. □

Most of the results for an LM-matrix can be carried over to those for a mixed matrix by way of the correspondence (3). In particular, we define an admissible transformation for a mixed matrix A to be a transformation of the form: $S\ A\ P_{\mathrm{c}}$, where S is a nonsingular matrix over $\boldsymbol{K}$ and P_{c} a permutation matrix. See [28] and [29].

6 Algorithm for CCF

An efficient algorithm is described here which computes the CCF of an LM-matrix $A \in \mathrm{LM}(\boldsymbol{F}/\boldsymbol{K}\,; m_Q, m_T, n)$ in $\mathrm{O}(n^3 \log n)$ time with arithmetic operations in the subfield $\boldsymbol{K}$ only, where $m = m_Q + m_T = \mathrm{O}(n)$ is assumed for simplicity in this complexity bound. This section is an improved presentation of §3.2 of Murota [38].

In order to illustrate a connection between the CCF and the Dulmage-Mendelsohn decomposition we first restrict ourselves to a nonsingular LM-matrix A. In this case the CCF can be found as follows.

[Algorithm (outline) for the CCF of a nonsingular A]

Step 1: Find $J \subseteq C$ such that both $Q[R_Q, J]$ and $T[R_T, C - J]$ are nonsingular (such J exists by Lemma 3).

Step 2: Let S denote the inverse of $Q[R_Q, J]$ and put

$$A' := \begin{pmatrix} S & O \\ O & I \end{pmatrix} A.$$

Step 3: Find the Dulmage-Mendelsohn decomposition $\bar{A}$ of A', namely, $\bar{A} := P_{\mathrm{r}}\ A'\ P_{\mathrm{c}}$ with suitable permutation matrices P_{r} and P_{c}. ($\bar{A}$ is the CCF of A.) □

The first step (Step 1) is nothing but the well-studied problem of matroid partition and a number of efficient algorithms are available for it; see Edmonds [9] and Lawler [20]. The DM-decomposition in the last step (Step 3) can be computed by first finding a maximum (perfect) matching in the bipartite graph associated with A', i.e., the graph denoted as $G(A')$ at the beginning of §3.1, and then decomposing an auxiliary digraph into strongly connected components. See, e.g., [5], [21], [28] for more detail on the DM-decomposition.

For the LM-matrix of Example 3, which is nonsingular, we can take $J = \{\xi_5, \xi_3, \xi_4, \eta_4, \eta_3\}$ in Step 1. The transformation matrix S given in Example 3 is equal to the inverse of

$$Q[R_Q, J] = \begin{array}{c} \begin{array}{ccccc} \xi_5 & \xi_3 & \xi_4 & \eta_4 & \eta_3 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix} \end{array}.$$

For a general (not necessarily nonsingular) LM-matrix it has been shown that the CCF can be constructed by identifying the minimum cuts in an independent-flow problem. See Prop. 20.1 of [28] as well as [41] for this reduction and Fujishige [13], [15] for independent-flow problems.

The detail of the algorithm for a general LM-matrix $A \in \mathrm{LM}(\boldsymbol{F}/\boldsymbol{K}\,; m_Q, m_T, n)$ is now described. As before let $R_T = \mathrm{Row}(T)$ and $C = \mathrm{Col}(A)$. Furthermore let C_Q be a disjoint copy of C, where the copy of $j \in C$ will be denoted as $j_Q \in C_Q$. The algorithm works with a directed graph $G = (V, B)$ with vertex set $V = R_T \cup C_Q \cup C$ and arc set $B = B_T \cup B_C \cup B^+ \cup M$, where

$$B_T = \{(i,j) \mid i \in R_T, j \in C, T_{ij} \neq 0\}, \quad B_C = \{(j_Q, j) \mid j \in C\},$$

and B^+ and M are sets of arcs which are defined and updated in the algorithm; B^+ consists of arcs from C_Q to C_Q and M from C to $R_T \cup C_Q$. The set of end-vertices of M (vertices incident to an arc in M) will be designated as ∂M ($\subseteq V$). Besides the graph G we use two matrices (or two-dimensional arrays) P and S, as well as a vector (or one-dimensional array) *base*. The array P represents a matrix over $\boldsymbol{K}$, of size $m_Q \times n$, where $P = Q$ at the beginning of the algorithm (Step 1 below). The other array S is also a matrix over $\boldsymbol{K}$, of size $m_Q \times m_Q$, which is set to the unit matrix in Step 1 and finally gives the matrix S in the admissible transformation (4). Variable *base* is a vector of size m_Q, which represents a mapping (correspondence): $R_Q \to C \cup \{0\}$.

[Algorithm for the CCF of a general A **]**

```
Step 1:
    $M := \emptyset$;   $base[i] := 0 \ (i \in R_Q)$;   $P[i,j] := Q_{ij} \ (i \in R_Q, j \in C)$;
    $S :=$ unit matrix of order $m_Q$.

Step 2:
    $I := \{i \in C \mid i_Q \in \partial M \cap C_Q\}$;
    $J := \{j \in C - I \mid \text{For all } i,\ base[i] = 0 \text{ implies } P[i,j] = 0\}$;
    $S_T^+ := R_T - \partial M$;   $S_Q^+ := \{j_Q \in C_Q \mid j \in C - (I \cup J)\}$;   $S^+ := S_T^+ \cup S_Q^+$;
    $S^- := C - \partial M$;
    $B^+ := \{(i_Q, j_Q) \mid h \in R_Q, j \in J, P[h,j] \neq 0, i = base[h]\}$;
    If there exists in $G$ a directed path from $S^+$ to $S^-$ then go to Step 3; otherwise
    (including the case where $S^+ = \emptyset$ or $S^- = \emptyset$) go to Step 4.

Step 3:
    Let $L$ ($\subseteq B$) be (the set of arcs on) a shortest path from $S^+$ to $S^-$ ("shortest" in
    the number of arcs);
    $M := (M - L) \cup \{(j,i) \mid (i,j) \in L \cap B_T\} \cup \{(j, j_Q) \mid (j_Q, j) \in L \cap B_C\}$;
    If the initial vertex ($\in S^+$) of the path $L$ belongs to $S_Q^+$, then do the following:
        {Let $j_Q$ ($\in S_Q^+ \subseteq C_Q$) be the initial vertex;
         Find $h$ such that $base[h] = 0$ and $P[h,j] \neq 0$;
                                                  [$j \in C$ corresponds to $j_Q \in C_Q$]
         $base[h] := j$;   $w := 1/P[h,j]$;
         $P[k,l] := P[k,l] - w \times P[k,j] \times P[h,l] \quad (h \neq k \in R_Q, l \in C)$;
         $S[k,l] := S[k,l] - w \times P[k,j] \times S[h,l] \quad (h \neq k \in R_Q, l \in R_Q)$ };
    For all $(i_Q, j_Q) \in L \cap B^+$ (in the order from $S^+$ to $S^-$ along $L$) do the following:
        {Find $h$ such that $i = base[h]$;        [$j \in C$ corresponds to $j_Q \in C_Q$]
         $base[h] := j$;   $w := 1/P[h,j]$;
         $P[k,l] := P[k,l] - w \times P[k,j] \times P[h,l] \quad (h \neq k \in R_Q, l \in C)$;
         $S[k,l] := S[k,l] - w \times P[k,j] \times S[h,l] \quad (h \neq k \in R_Q, l \in R_Q)$ };
```

Go to Step 2.

Step 4:

Let V_∞ ($\subseteq V$) be the set of vertices reachable from S^+ by a directed path in G;

Let V_0 ($\subseteq V$) be the set of vertices reachable to S^- by a directed path in G;

$C_0 := C \cap V_0; \quad C_\infty := C \cap V_\infty;$

Let G' denote the graph obtained from G by deleting the vertices $V_0 \cup V_\infty$ (and arcs incident thereto);

Decompose G' into strongly connected components $\{V_\lambda \mid \lambda \in \Lambda\}$ ($V_\lambda \subseteq V$);

Let $\{C_k \mid k = 1, \ldots, b\}$ be the subcollection of $\{C \cap V_\lambda \mid \lambda \in \Lambda\}$ consisting of all the nonempty sets $C \cap V_\lambda$, where C_k's are indexed in such a way that for $l < k$ there does not exist a directed path in G' from C_k to C_l;

$R_0 := (R_T \cap V_0) \cup \{h \in R_Q \mid \mathit{base}[h] \in C_0\};$

$R_\infty := (R_T \cap V_\infty) \cup \{h \in R_Q \mid \mathit{base}[h] \in C_\infty \cup \{0\}\};$

$R_k := (R_T \cap V_k) \cup \{h \in R_Q \mid \mathit{base}[h] \in C_k\}$ $(k = 1, \ldots, b);$

$\bar{A} := P_{\mathrm{r}} \begin{pmatrix} P \\ T \end{pmatrix} P_{\mathrm{c}}$, where the permutation matrices P_{r} and P_{c} are determined so that the rows and the columns of $\bar{A}$ are ordered as $(R_0; R_1, \ldots, R_b; R_\infty)$ and $(C_0; C_1, \ldots, C_b; C_\infty)$, respectively. □

The subsets $I \subseteq C$ and $J \subseteq C$ represent the structure of the matroid $\boldsymbol{M}(Q)$ defined by the matrix Q; I is an independent set in $\boldsymbol{M}(Q)$ (i.e., $\operatorname{rank} Q[R_Q, I] = |I|$), whereas $J \cup I$ is the closure (cf. [53], [54]) of I (i.e., $J = \{j \in C - I \mid \operatorname{rank} Q[R_Q, I \cup \{j\}] = |I|\}$). On the other hand, $(\partial M \cap C) - I$ is an independent set in the other matroid $\boldsymbol{M}(T)$ defined by the matrix T. Hence $\partial M \cap C$ $(= I \cup ((\partial M \cap C) - I)$) is independent in $\boldsymbol{M}(Q) \vee \boldsymbol{M}(T)$. Since $\boldsymbol{M}(Q) \vee \boldsymbol{M}(T) = \boldsymbol{M}(A)$, by Lemma 4, we have $\operatorname{rank} A[R, \partial M \cap C] = |M|$. At each execution of Step 3 the size of M increases by one, and at the termination of the algorithm we have the relation: $\operatorname{rank} A = |M|$.

The matrix $\bar{A}$ is the CCF of the input matrix A, where $\{R_0; R_1, \ldots, R_b; R_\infty\}$ and $\{C_0; C_1, \ldots, C_b; C_\infty\}$ give the partitions of the row set and the column set, respectively. The partial order among the blocks is induced from the partial order among the strongly connected components $\{V_\lambda \mid \lambda \in \Lambda\}$.

The shortest path in Step 3 and the strongly connected components in Step 4 can be found in time linear in the size of the graph G, which is $\mathrm{O}((n+m)^2)$, by means of the standard graph algorithms; see, e.g., [1].

The updates of P in Step 3 are the standard pivoting operations [16] on P, which is a matrix over the subfield $\boldsymbol{K}$. The sparsity of P should be taken into account in actual implementations of the algorithm; for example, $P[h, j] = 0$ if $\mathit{base}[h] = 0$ and $j \in I \cup J$. Computational techniques developed for solving sparse linear programs can be utilized here. As indicated in Step 3, pivoting operations are required for each arc $(i_Q, j_Q) \in L \cap B^+$. It is important to traverse the path L from S^+ to S^-, not from S^- to S^+, to avoid unnecessary fill-ins. When the transformation matrix S is not needed, it may simply be eliminated from the computation without any side effect.

The above algorithm will be efficient enough also for practical applications. It would be still more efficient if we first compute the DM-decomposition by purely graph-theoretic algorithm and then apply the above algorithm to each of the DM-irreducible components; such two-stage procedure works since the CCF is a refinement of the DM-decomposition.

Finally we mention a characterization of the LM-irreducibility in terms of the graph used in the algorithm.

Theorem 17 *Let A be a square LM-matrix. A is LM-irreducible (and hence nonsingular) if and only if in Step 4 of the algorithm both V_0 and V_∞ are empty and graph G' $(= G)$ is strongly connected.* □

Example 7 The algorithm above is illustrated here for a 4×5 LM-matrix $A = \binom{Q}{T}$ with

$$Q = \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 2 & 1 & 1 & 0 \end{pmatrix} \end{array}, \quad T = \begin{array}{c|ccccc} & x_1 & x_2 & x_3 & x_4 & x_5 \\ \hline f_1 & t_1 & 0 & 0 & 0 & t_2 \\ f_2 & 0 & t_3 & 0 & 0 & t_4 \end{array},$$

where $\mathrm{Col}(A) = C = \{x_1, x_2, x_3, x_4, x_5\}$ and $\mathrm{Row}(T) = R_T = \{f_1, f_2\}$. We work with a 2×5 matrix P, a 2×2 matrix S, and a vector *base* of size 2. The copy of C is denoted as $C_Q = \{x_{1Q}, x_{2Q}, x_{3Q}, x_{4Q}, x_{5Q}\}$.

The flow of computation is traced below.

Step 1: $M := \emptyset$;

$$\mathit{base} := \begin{array}{c|c} r_1 & 0 \\ r_2 & 0 \end{array}, \quad P := \begin{array}{c|ccccc} & x_1 & x_2 & x_3 & x_4 & x_5 \\ \hline r_1 & 1 & 1 & 1 & 1 & 0 \\ r_2 & 0 & 2 & 1 & 1 & 0 \end{array}, \quad S := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Step 2: $I := \emptyset$; $J := \{x_5\}$;

$S_T^+ := \{f_1, f_2\}$; $S_Q^+ := \{x_{1Q}, x_{2Q}, x_{3Q}, x_{4Q}\}$; $S^+ := \{f_1, f_2, x_{1Q}, x_{2Q}, x_{3Q}, x_{4Q}\}$; $S^- := \{x_1, x_2, x_3, x_4, x_5\}$;

$B^+ := \emptyset$;

There exists a path from S^+ to S^-. [See $G^{(0)}$ in Fig.1]

Step 3: $L := \{(x_{1Q}, x_1)\}$; $M := \{(x_1, x_{1Q})\}$;

The initial vertex x_{1Q} of L is in S_Q^+, and the matrices are updated (with $h = r_1$) to

$$\mathit{base} := \begin{array}{c|c} r_1 & x_1 \\ r_2 & 0 \end{array}, \quad P := \begin{array}{c|ccccc} & x_1 & x_2 & x_3 & x_4 & x_5 \\ \hline r_1 & 1 & 1 & 1 & 1 & 0 \\ r_2 & 0 & 2 & 1 & 1 & 0 \end{array}, \quad S := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Noting $L \cap B^+ = \emptyset$ we return to Step 2.

Step 2: $I := \{x_1\}$; $J := \{x_5\}$;

$S_T^+ := \{f_1, f_2\}$; $S_Q^+ := \{x_{2Q}, x_{3Q}, x_{4Q}\}$; $S^+ := \{f_1, f_2, x_{2Q}, x_{3Q}, x_{4Q}\}$;

$S^- := \{x_2, x_3, x_4, x_5\}$;

$B^+ := \emptyset$;

There exists a path from S^+ to S^-. [See $G^{(1)}$ in Fig.2]

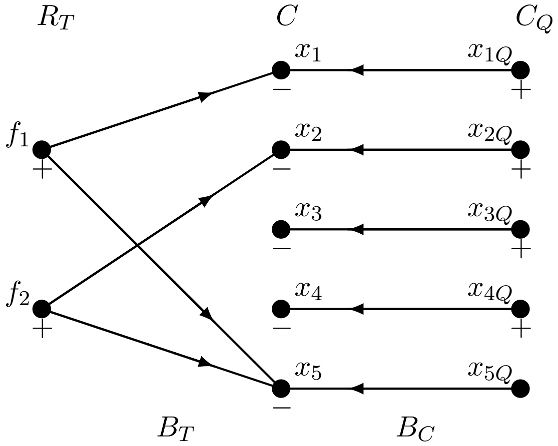

Figure 1: Graph $G^{(0)}$ (+: vertices in S^+; −: vertices in S^-)

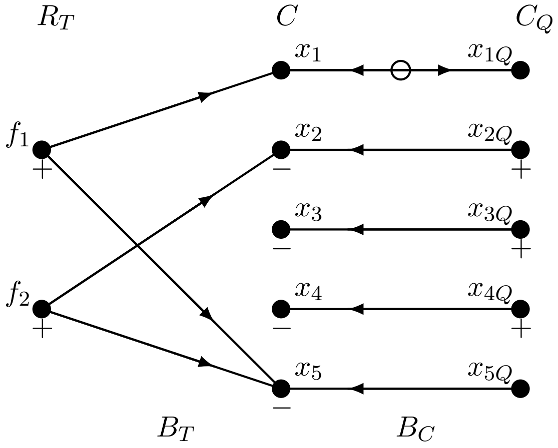

Figure 2: Graph $G^{(1)}$ (○: arc in M; +: vertices in S^+; −: vertices in S^-)

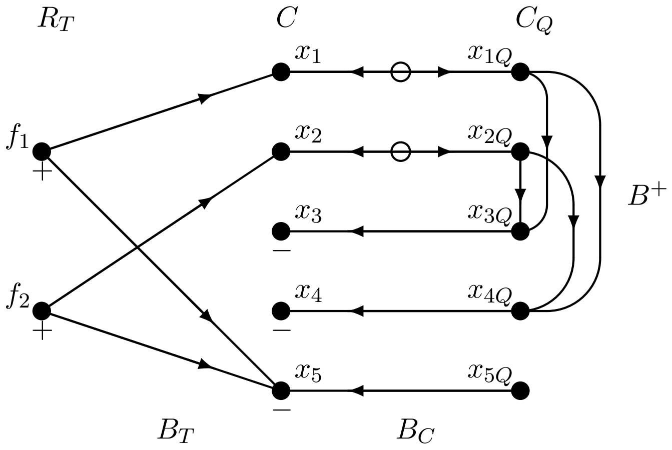

Figure 3: Graph $G^{(2)}$ (◯: arcs in M; $+$: vertices in S^+; $-$: vertices in S^-)

Step 3: $L := \{(x_{2Q}, x_2)\}$; $M := \{(x_1, x_{1Q}), (x_2, x_{2Q})\}$;

The initial vertex x_{2Q} of L is in S_Q^+, and the matrices are updated (with $h = r_2$) to

$$base := \begin{array}{c} r_1 \\ r_2 \end{array} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad P := \begin{array}{c|ccccc} & x_1 & x_2 & x_3 & x_4 & x_5 \\ \hline r_1 & 1 & 0 & 1/2 & 1/2 & 0 \\ r_2 & 0 & 2 & 1 & 1 & 0 \end{array}, \quad S := \begin{pmatrix} 1 & -1/2 \\ 0 & 1 \end{pmatrix}.$$

Noting $L \cap B^+ = \emptyset$ we return to Step 2.

Step 2: $I := \{x_1, x_2\}$; $J := \{x_3, x_4, x_5\}$;

$S_T^+ := \{f_1, f_2\}$; $S_Q^+ := \emptyset$; $S^+ := \{f_1, f_2\}$; $S^- := \{x_3, x_4, x_5\}$;

$B^+ := \{(x_{1Q}, x_{3Q}), (x_{1Q}, x_{4Q}), (x_{2Q}, x_{3Q}), (x_{2Q}, x_{4Q})\}$;

There exists a path from S^+ to S^-. [See $G^{(2)}$ in Fig.3]

Step 3: $L := \{(f_1, x_5)\}$; $M := \{(x_1, x_{1Q}), (x_2, x_{2Q}), (x_5, f_1)\}$;

The initial vertex $f_1 \notin S_Q^+$ and $L \cap B^+ = \emptyset$, and therefore the matrices remain unchanged and we return to Step 2.

Step 2: $I := \{x_1, x_2\}$; $J := \{x_3, x_4, x_5\}$;

$S_T^+ := \{f_2\}$; $S_Q^+ := \emptyset$; $S^+ := \{f_2\}$; $S^- := \{x_3, x_4\}$;

$B^+ := \{(x_{1Q}, x_{3Q}), (x_{1Q}, x_{4Q}), (x_{2Q}, x_{3Q}), (x_{2Q}, x_{4Q})\}$;

There exists a path from S^+ to S^-. [See $G^{(3)}$ in Fig.4]

Step 3: $L := \{(f_2, x_2), (x_2, x_{2Q}), (x_{2Q}, x_{3Q}), (x_{3Q}, x_3)\}$;

$M := \{(x_1, x_{1Q}), (x_3, x_{3Q}), (x_5, f_1), (x_2, f_2)\}$;

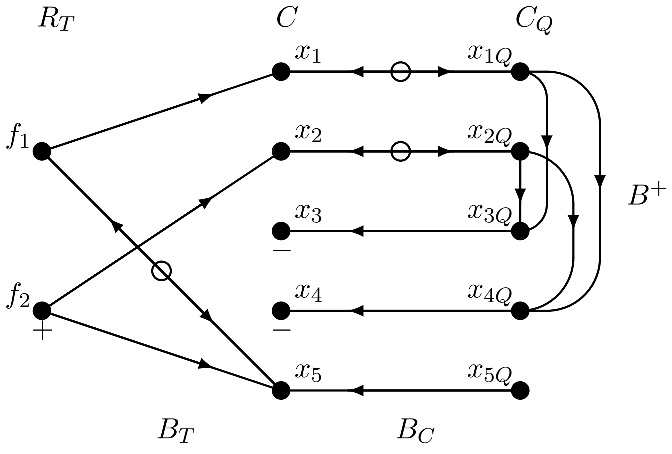

Figure 4: Graph $G^{(3)}$ (◯: arcs in M; $+$: vertex in S^+; $-$: vertices in S^-)

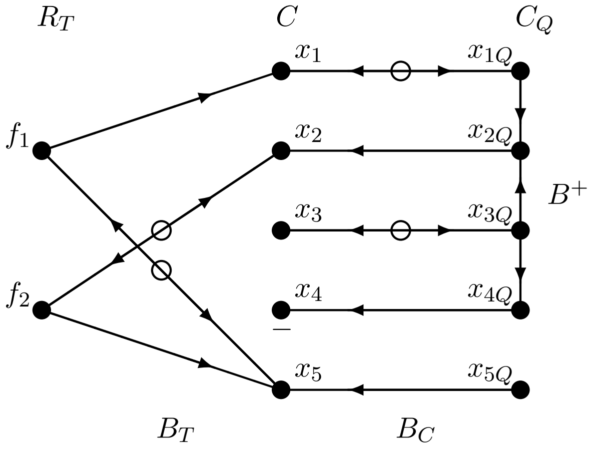

Figure 5: Graph $G^{(4)}$ (◯: arcs in M; $S^+ = \emptyset$, $-$: vertex in S^-)

The initial vertex $f_2 \notin S_Q^+$ and $L \cap B^+ = \{(x_{2Q}, x_{3Q})\}$, and the matrices are updated (with $h = r_2$) to

$$\mathit{base} := \begin{array}{c} r_1 \\ r_2 \end{array}\begin{pmatrix} x_1 \\ x_3 \end{pmatrix}, \quad P := \begin{array}{c} \\ r_1 \\ r_2 \end{array}\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 \end{pmatrix} \end{array}, \quad S := \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Step 2: $I := \{x_1, x_3\}$; $J := \{x_2, x_4, x_5\}$;

$S_T^+ := \emptyset$; $S_Q^+ := \emptyset$; $S^+ := \emptyset$; $S^- := \{x_4\}$;

$B^+ := \{(x_{1Q}, x_{2Q}), (x_{3Q}, x_{2Q}), (x_{3Q}, x_{4Q})\}$;

There exists no path from S^+ $(= \emptyset)$ to S^-. [See $G^{(4)}$ in Fig.5]

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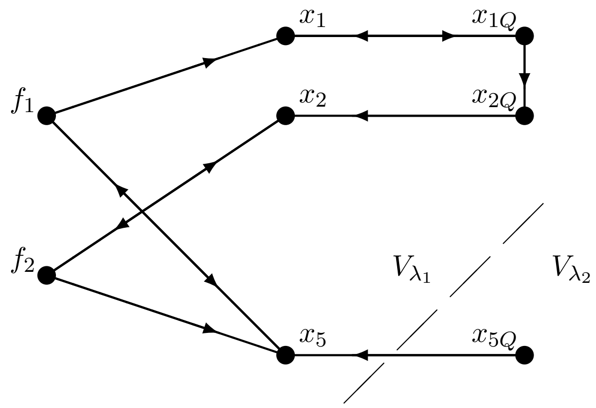

Figure 6: Graph G' $\quad$ (V_{λ_1}, V_{λ_2}: strongly connected components)

Step 4: $V_\infty := \emptyset$; $V_0 := \{x_3, x_4, x_{3Q}, x_{4Q}\}$;

$C_0 := \{x_3, x_4\}$; $C_\infty := \emptyset$;

Strongly connected components of G' (cf. Fig.6) are given by $\{V_{\lambda_1}, V_{\lambda_2}\}$, where $V_{\lambda_1} = \{x_1, x_2, x_5, x_{1Q}, x_{2Q}, f_1, f_2\}$ and $V_{\lambda_2} = \{x_{5Q}\}$;

Since $C \cap V_{\lambda_2} = \emptyset$, we have $b := 1$ and $C_1 := C \cap V_{\lambda_1} = \{x_1, x_2, x_5\}$;

$R_0 := \{r_2\}$; $R_\infty := \emptyset$; $R_1 := \{r_1, f_1, f_2\}$;

$$\bar{A} := P_{\rm r} \begin{pmatrix} P \\ T \end{pmatrix} P_{\rm c} = \begin{array}{c} \\ r_2 \\ r_1 \\ f_1 \\ f_2 \end{array} \begin{array}{c} \begin{array}{ccccc} x_3 & x_4 & x_1 & x_2 & x_5 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 2 & 0 \\ & & 1 & -1 & 0 \\ & & t_1 & 0 & t_2 \\ & & 0 & t_3 & t_4 \end{pmatrix} \end{array}$$

is the CCF. □

7 Conclusion

As a mathematical model for investigating the structure of linear dynamical systems, Murota [25], [28] proposed to consider a polynomial matrix $D(s)$ in indeterminate s over a field $\boldsymbol{F}(\supset \mathbf{Q})$ which is represented as

$$D(s) = Q(s) + T(s),$$

where

(A1): The set of the nonzero coefficients of the entries of $T(s)$ is algebraically independent over $\mathbf{Q}$, and

(A2): Every nonvanishing subdeterminant of $Q(s)$ is a monomial in s over $\mathbf{Q}$.

Note that the assumption (A1) implies $D(s)$ is a mixed matrix with base field $\boldsymbol{K} = \mathbf{Q}(s)$. See [25], [28], [38] for physical backgrounds of the conditions (A1) and (A2); [27], [31], [42] for applications to control problems; and [30], [35], [39] for more recent results on such polynomial matrices.

As an extension of the CCF, Murota [34] considered the decomposition of an LM-matrix $A \in \mathrm{LM}(\boldsymbol{F}/\boldsymbol{K})$ with respect to a larger class of admissible transformations of the form: $S_{\mathrm{r}}\ A\ S_{\mathrm{c}}$ with S_{r} and S_{c} nonsingular matrices over $\boldsymbol{K}$. This paper also considered the decomposition of A under this extended admissible transformation when A has certain symmetry expressed as an invariance with respect to a finite group.

Poljak [45] gave a combinatorial characterization to the rank of a power product, $\mathrm{rank}\, T^k$, for a generic matrix T. It will be interesting to see whether his result can be extended to a mixed matrix.

Yamada-Luenberger [57] introduced the notion of "column-structured matrices" as a generalization of generic matrices.

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