Book review on

Discrete Convex Analysis

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Discrete Convex Analysis

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This monograph brings together continuous optimization and discrete optimization. More precisely, it establishes a new theory of nonlinear discrete optimization by importing techniques of continuous optimization, in particular convex analysis. On the other hand, viewed from the continuous side, it proposes a theory of convex functions with some combinatorial properties. The theory has been developed mainly by the author since 1995. It provides many new insights and has already led to interesting applications.

While discrete optimization problems are generally NP-hard, several very important types of problems have been solved well, in the sense of optimality conditions, duality relations, and—most important—efficient polynomial-time algorithms. This area, known as *combinatorial optimization*, has been very fruitful in the last fifty years.

However, many results in combinatorial optimization are problem-specific. A special combinatorial structure is analyzed, which then leads to algorithms for optimizing (usually linear) objective functions over this structure. Examples are network flow algorithms, algorithms for matching shortest paths or optimum spanning trees in graphs. In some cases, the reason why some structures are easy to deal with and others are not could be explained by the theory of matroids and submodular functions: we know how to optimize linear objective functions over discrete sets defined by submodular functions.

Submodular functions, i.e., real functions f defined on the power set of a finite set U satisfying $f(X \cup Y) + f(X \cap Y) \leq f(X) + f(Y)$ for all $X, Y \subseteq U$, have nice properties, similar to convex functions in the continuous world. Indeed, Lovász proved the important characterization that a discrete set function is submodular if and only if its so-called Lovász extension (sometimes also called Choquet integral) is convex. Frank showed a sandwich theorem: just as for a concave function f and a convex function h with $f \leq h$ there is always a linear function g with $f \leq g \leq h$, the same holds if we replace convex by submodular and concave by supermodular (f is called supermodular if $-f$ is submodular). Moreover the linear function g can be chosen integer-valued if f and h are. This can be regarded as equivalent to an earlier theorem by Edmonds. A discrete analogue of Fenchel duality was proved by Fujishige, who also wrote the standard reference for submodular functions (at least up to the appearance of Murota's book).

If we identify subsets of an n-element ground set by their characteristic vectors, submodular functions can be viewed as functions $f : \{0,1\}^n \to \mathbb{R}$ with $f(x \vee y) + f(x \wedge y) \leq f(x) + f(y)$ for all $x, y \in \{0,1\}^n$, where $\vee$ and $\wedge$ denote component-wise maximum and minimum, respectively. They are now generalized as follows: A function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{\infty\}$ is called L-convex if there exists an $r \in \mathbb{R}$ with $f(x + (1, \ldots, 1)) = f(x) + r$ for all $x \in \mathbb{Z}^n$, and $f(x \vee y) + f(x \wedge y) \leq f(x) + f(y)$ for all $x, y \in \mathbb{Z}^n$. The second class of functions considered are also natural discrete analogues of convex functions: A function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{\infty\}$ is called M-convex if for all $x, y \in \mathbb{Z}^n$ with $f(x), f(y) < \infty$ and for each index i with $x_i > y_i$ there is an index j with $x_j < y_j$ and $f(x - e_i + e_j) + f(y + e_i - e_j) \leq f(x) + f(y)$, where e_i denotes the ith unit vector. L-convex and M-convex functions are conjugate to each other by a discrete analogue of the Legendre-Fenchel transformation.

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Although there were a few isolated results on nonlinear discrete optimization problems, there was no unified theory. Discrete convex analysis provides such a theory. L-convex and M-convex functions are the main subject. One can view it from two sides: it generalizes the theory of (poly)matroids and submodular functions, but it can also be viewed as a theory of convex functions with some combinatorial properties.

This book provides a theory generalizing many important results and yielding new insights on them. It will be interesting for researchers in continuous optimization, in particular convex analysis, as well as for anybody interested in discrete optimization.

The book is exceptionally well conceived. It is completely self-contained and written very carefully. There are only very few monographs of such outstanding quality. Although the material is quite difficult and the book contains many deep results, it can be read and understood by nonspecialists and can be used for advanced graduate courses. The author invested a lot of effort in the excellent introductory chapter to acquaint the reader with the theory to come and to create interest in the book. Consequently, this book should be read from the beginning!

After the introductory part, the author reviews fundamental concepts. Then he introduces M-convex sets, L-convex sets, M-convex functions, and L-convex functions in detail, generalizing known concepts. A central chapter deals with conjugacy and duality. The book then continues with applications to (nonlinear) network flows, minimizing M-convex and L-convex functions (related to minimizing submodular functions, where the author reviews recent breakthroughs) and generalizations of submodular flows. The concluding chapters contain applications to economics (equilibria theory) and engineering (systems analysis). Relations to matrix theory and electrical networks may also be interesting for many readers.

The material has not appeared in any English book before, parts have been published in a Japanese book by the same author, and in journal papers, mainly by Murota himself, some with Japanese colleagues. This important theory is now accessible, which will lead to a better understanding of the subject.

No mathematical library should be without this book. Everyone interested in optimization should consider it, but also researchers from related fields will benefit from reading (parts of) Murota's book. It needs some time to understand the beauty of the theory, but it is worth spending this time as it leads to a better understanding of classical and new concepts. I recommend the book warmly.

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