# Preface

"Discrete Convex Analysis" is aimed at establishing a novel theoretical framework for solvable discrete optimization problems by means of a combination of the ideas in continuous optimization and combinatorial optimization. The theoretical framework of convex analysis is adapted to discrete settings and the mathematical results in matroid/submodular function theory are generalized. Viewed from the continuous side, the theory can be classified as a theory of convex functions $f : \mathbf{R}^n \rightarrow \mathbf{R}$ that have additional combinatorial properties. Viewed from the discrete side, it is a theory of discrete functions $f : \mathbf{Z}^n \rightarrow \mathbf{Z}$ that enjoy certain nice properties comparable to convexity. Symbolically,

$$\text{Discrete Convex Analysis} \ = \ \text{Convex Analysis} \ + \ \text{Matroid Theory}.$$

The theory puts emphasis on duality and conjugacy as well as on algorithms. This results in a novel duality framework for nonlinear integer programming.

Two convexity concepts, called L-convexity and M-convexity, play primary roles, where "L" stands for "Lattice" and "M" for "Matroid." L-convex functions and M-convex functions are convex functions with additional combinatorial properties distinguished by "L" and "M," and they are conjugate to each other through a discrete version of the Legendre–Fenchel transformation. L-convex functions and M-convex functions generalize, respectively, the concepts of submodular set functions and base polyhedra of (poly)matroids.

L-convexity and M-convexity prevail in discrete systems.

- In network flow problems, flow and tension are dual objects. Roughly speaking, flow corresponds to M-convexity and tension to L-convexity.
- In matroids, the rank function corresponds to L-convexity and the base family to M-convexity.
- M-matrices in matrix theory correspond to L-convexity, and their inverses to M-convexity. Hence, in a discretization of the Poisson problem of partial differential equations, for example, the differential operator corresponds to L-convexity and the Green function to M-convexity.
- Dirichlet forms in probability theory are essentially the same as quadratic L-convex functions.

The present book is intended to be read profitably by graduate students in operations research, mathematics, and computer science, and also by mathematicsoriented practitioners and application-oriented mathematicians. Self-contained presentation is envisaged. In particular, no familiarity with matroid theory nor with convex analysis is assumed. On the contrary, the reader will hopefully acquire a unified view on matroids and convex functions through a variety of examples of discrete systems and the axiomatic approach presented in this book.

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