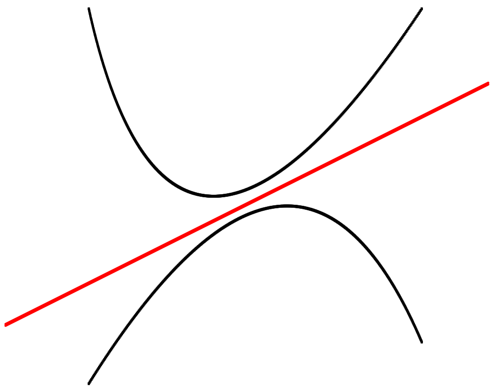
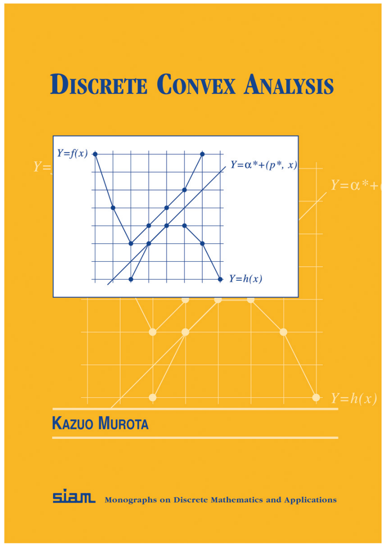
Modern Aspects of Submodularity, Georgia Tech, March 19, 2012

Introduction to
Discrete Convex Analysis

Kazuo Murota (U. Tokyo)





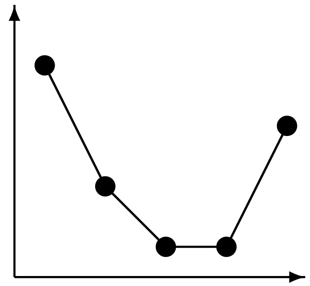
120319atlanta1Intro

Discrete Convex Analysis

Convexity Paradigm in Discrete Optimization

Matroid Theory + Convex Analysis

Submodular fn	$\leftrightarrow$	Matroid base
$\Downarrow$		$\Downarrow$
L-convex fn	$\leftrightarrow$	M-convex fn



- Global optimality $\Longleftrightarrow$ local optimality
- Conjugacy: Legendre–Fenchel transform
- Duality (Fenchel min-max, discrete separation)
- Minimization algorithms
- Applications: OR, game, economics, matrices

Applications

- Combinatorial optimization
  - matching, even factor, min-cost flow, shortest path, min-cost tension
- Mathematical economics / Game theory
  - indivisible goods, stable marriage
- Operations research
  - inventory, queueing, resource allocation
- Discrete structures
  - finite metric space
- Algebra
  - polynomial matrix, tropical geometry

Some History

1935	Matroid	Whitney
1965	Submodular function	Edmonds
1975	Engrg application of matroid	Iri, Recski
1983	Submodularity and convexity	
		Lovász, Frank, Fujishige
1990	Valuated matroid	Dress–Wenzel
	Integrally convex fn	Favati–Tardella
1996	Discrete convex analysis	Murota
2000	Submod. fn minimization algorithm	
		Iwata–Fleischer–Fujishige, Schrijver

Contents

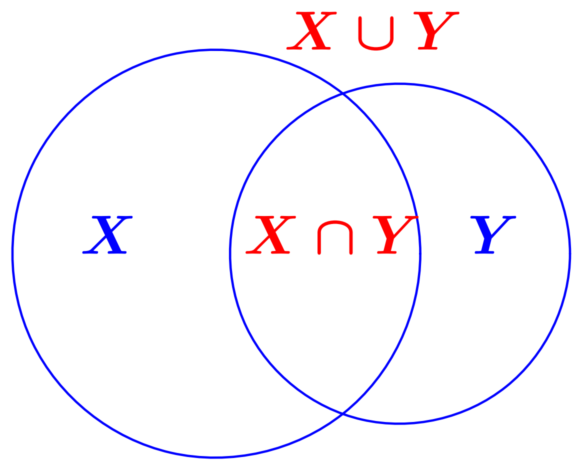
B1.

Submodularity and Convexity (1980's)

Submodular Function

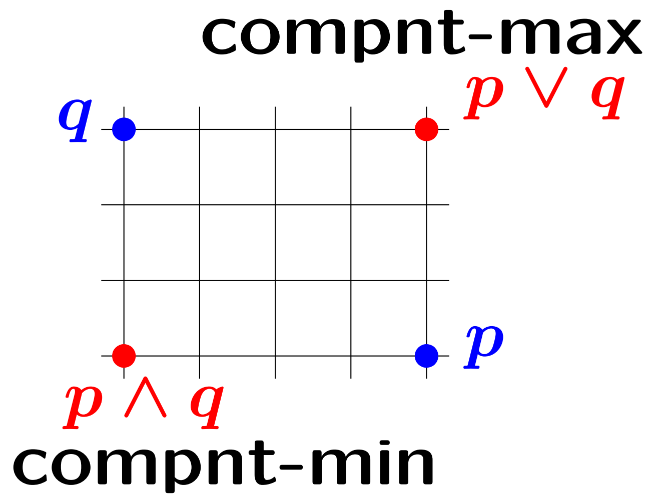
Set function ρ is submodular:

$$\rho(X) + \rho(Y) \geq \rho(X \cup Y) + \rho(X \cap Y)$$



$g : \mathbb{Z}^n \to \mathbb{R}$ is submodular:

$$g(p) + g(q) \geq g(p \vee q) + g(p \wedge q)$$



Submodularity & Convexity in 1980's

$$\rho(X) + \rho(Y) \geq \rho(X \cup Y) + \rho(X \cap Y)$$

- min/max algorithms (Grötschel–Lovász–Schrijver/ Jensen–Korte, Lovász)

  min $\Rightarrow$ polynomial, max $\Rightarrow$ NP-hard

- Convex extension (Lovász)

  set fn is submod $\Leftrightarrow$ Lovász ext is convex

- Duality theorems (Edmonds, Frank, Fujishige)

  discrete separation, Fenchel min-max

Duality for submodular set functions = Convexity + Discreteness

Frank's Discrete Separation

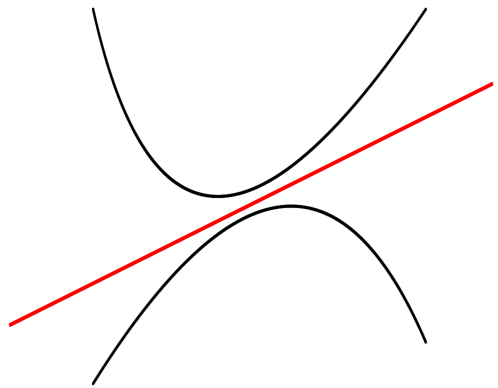
(Frank 82)

$\rho : 2^V \to \mathrm{R}$: submodular $\qquad (\rho(\emptyset) = 0)$

$\mu : 2^V \to \mathrm{R}$: supermodular $\qquad (\mu(\emptyset) = 0)$

- $\rho(X) \geq \mu(X) \quad (\forall X \subseteq V) \Rightarrow \exists\ x^* \in \mathrm{R}^V$:

$$\rho(X) \geq x^*(X) \geq \mu(X) \qquad (\forall X \subseteq V)$$



- ρ, μ: integer-valued $\Rightarrow x^* \in \mathrm{Z}^V$

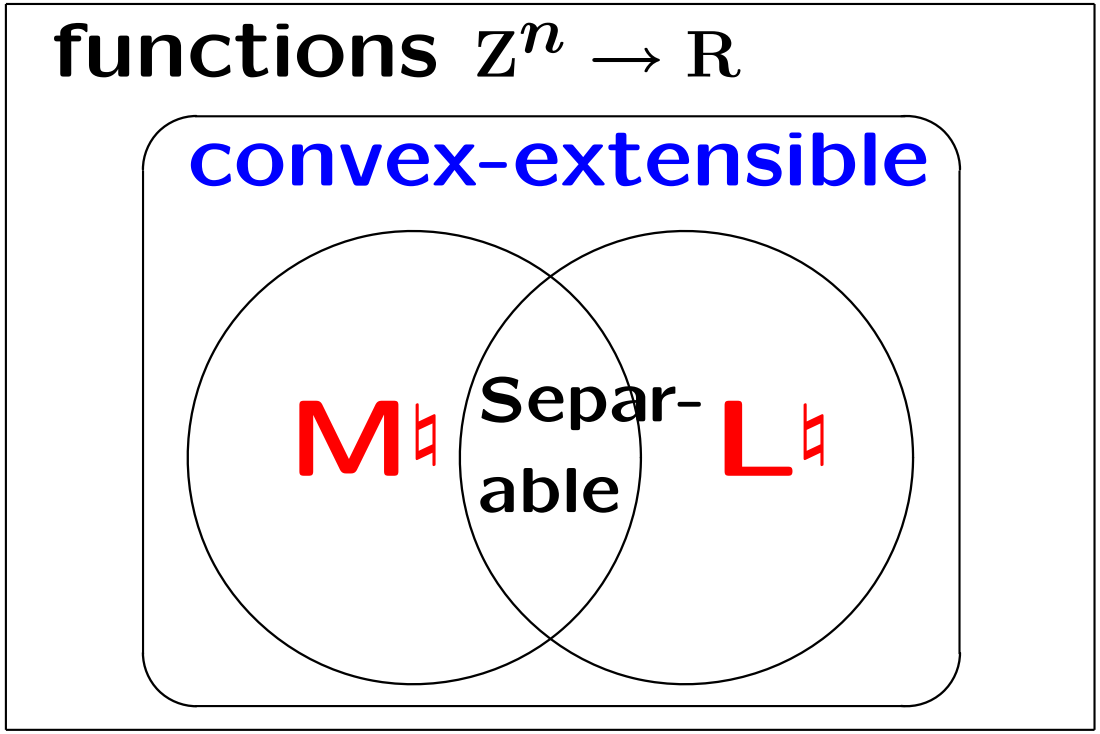
Equivalent to Edmonds' polymatroid intersection

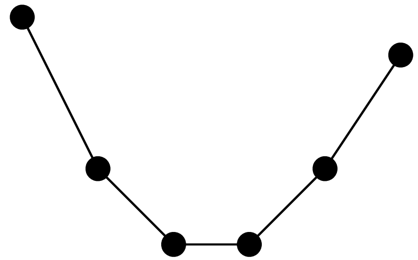
B2.

L-convex and M-convex Functions

Discrete Convex Functions

$$f : \mathbb{Z}^n \to \mathbb{R}$$





f is convex-extensible

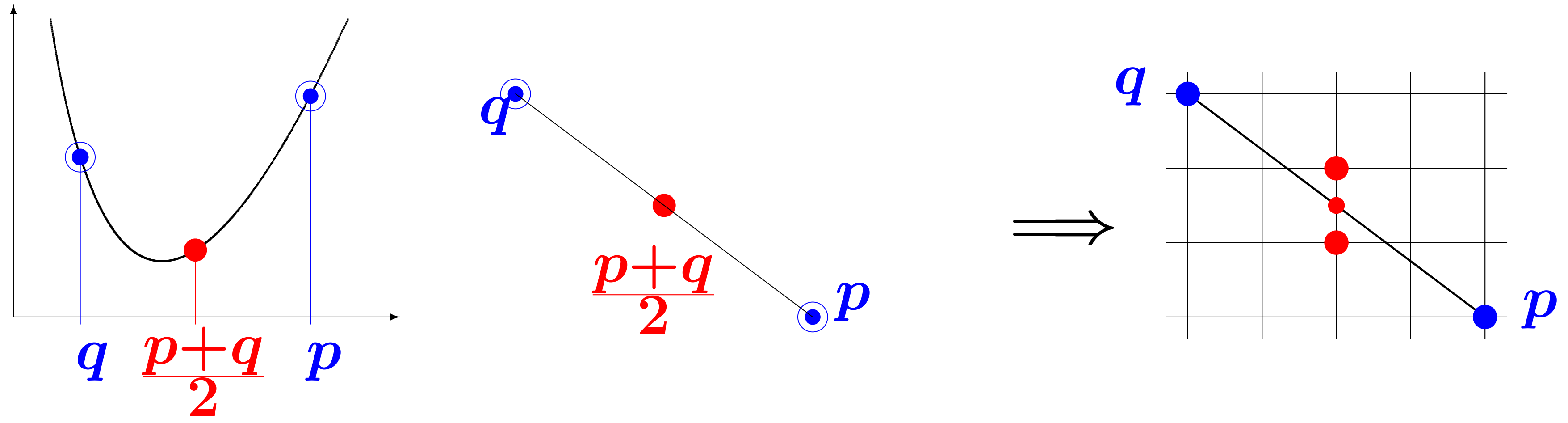
$\Leftrightarrow$ $\exists$ convex $\overline{f}$:

$$f(x) = \overline{f}(x)$$

Convex-extensibility does not help much

L♮-convexity from Mid-pt-convexity

(Murota 98, Fujishige–Murota 00)



Mid-point convex ($g : \mathrm{R}^n \to \mathrm{R}$)**:**

$$g(p) + g(q) \geq 2g\left(\frac{p+q}{2}\right)$$

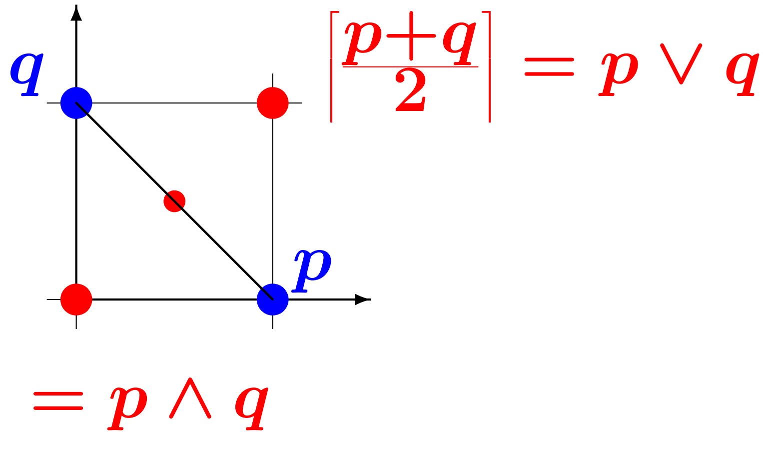
$\Longrightarrow$ **Discrete mid-point convex** ($g : \mathrm{Z}^n \to \mathrm{R}$)

$$g(p) + g(q) \geq g\left(\left\lceil \frac{p+q}{2} \right\rceil\right) + g\left(\left\lfloor \frac{p+q}{2} \right\rfloor\right)$$

L♮-convex function (L = Lattice)

Mid-pt Convexity for 01-Vectors

For $p, q \in \{0,1\}^n$



Discrete mid-pt convexity:

$$g(p) + g(q) \geq g\left(\left\lceil \frac{p+q}{2} \right\rceil\right) + g\left(\left\lfloor \frac{p+q}{2} \right\rfloor\right)$$

$\Longleftrightarrow$ **Submodularity:**

$$g(p) + g(q) \geq g(p \vee q) + g(p \wedge q)$$

L$^\natural$-convexity from Submodularity

—Original definition of L$^\natural$-convexity—

Def: $g : \mathbf{Z}^n \to \mathbf{R}$ is L$^\natural$-convex $\iff$

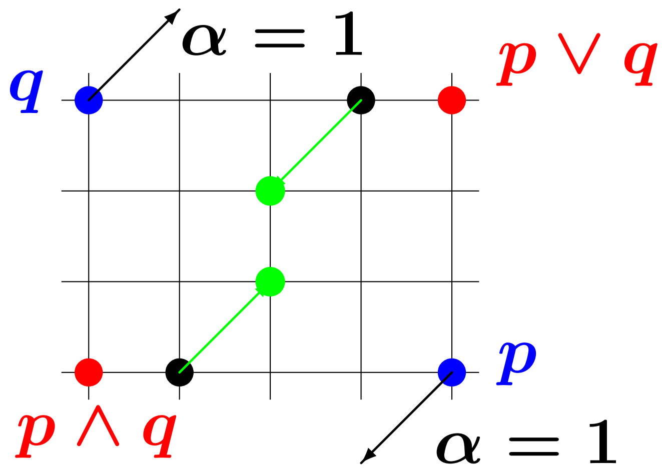
$\tilde{g}(p_0, p) = g(p - p_0 \mathbf{1})$ is submodular in (p_0, p)

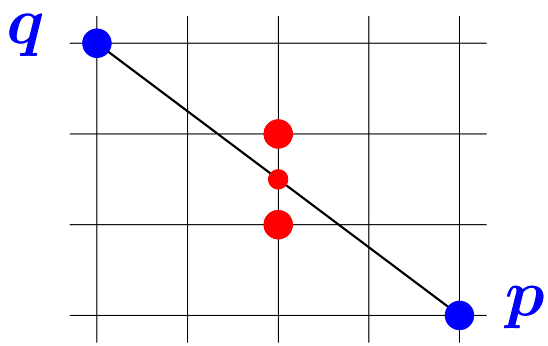
$\tilde{g} : \mathbf{Z}^{n+1} \to \mathbf{R}$, $\quad \mathbf{1} = (1, 1, \ldots, 1, 1)$

Translation Submodularity ($L^\natural$)

$$g(p) + g(q) \geq g((p - \alpha 1) \vee q) + g(p \wedge (q + \alpha 1))$$

$(\alpha \geq 0)$





discrete mid-pt convex

$\tilde{g}(p_0, p) = g(p - p_0 1)$ is submodular in (p_0, p)

$\Leftrightarrow$ translation submodular (Fujishige-Murota 00)

$\Leftrightarrow$ discrete mid-pt convex (Fujishige-Murota 00)

$\Leftrightarrow$ submod. integ. convex (Favati-Tardella 90)

L$^\natural$-convex vs Submodular

- L$^\natural$-convex function is convex-extensible
- General submodular function is NOT

Fact 1:

Any function $g : \mathbb{Z} \to \mathbb{R}$ is submodular

$\Rightarrow$ Submodularity does not guarantee convexity

Fact 2:

A function $g : \mathbb{Z} \to \mathbb{R}$ is L$^\natural$-convex

$\iff$ $g(p-1) + g(p+1) \geq 2g(p)$ for all $p \in \mathbb{Z}$

(This slide shows my answer to a question during the talk)

L$^\natural$-convex Function: Examples

Quadratic: $g(p) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} p_i p_j$ is L$^\natural$-convex

$$\Leftrightarrow a_{ij} \leq 0 \quad (i \neq j), \qquad \sum_{j=1}^{n} a_{ij} \geq 0 \ (\forall i)$$

Separable convex: For univariate convex ψ_i and ψ_{ij}

$$g(p) = \sum_{i=1}^{n} \psi_i(p_i) + \sum_{i \neq j} \psi_{ij}(p_i - p_j)$$

Range: $g(p) = \max\{p_1, p_2, \ldots, p_n\} - \min\{p_1, p_2, \ldots, p_n\}$

Submodular set function: $\rho : 2^V \to \overline{\mathrm{R}}$

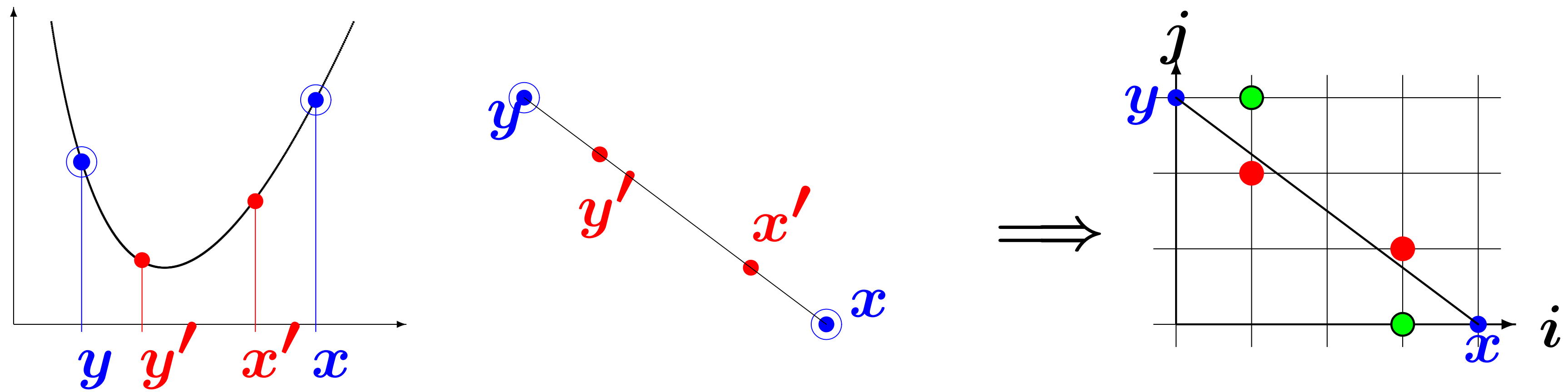
$$\Leftrightarrow \rho(X) = g(\chi_X) \text{ for L}^\natural\text{-convex } g$$

Multimodular: $h : \mathrm{Z}^n \to \overline{\mathrm{R}}$ is multimodular $\Leftrightarrow$

$h(p) = g(p_1, p_1 + p_2, \ldots, p_1 + \cdots + p_n)$ for L$^\natural$-convex g

M♮-convexity from Equi-dist-convexity

(Murota 96, Murota–Shioura 99)



Equi-distance convex ($f : \mathrm{R}^n \to \mathrm{R}$):

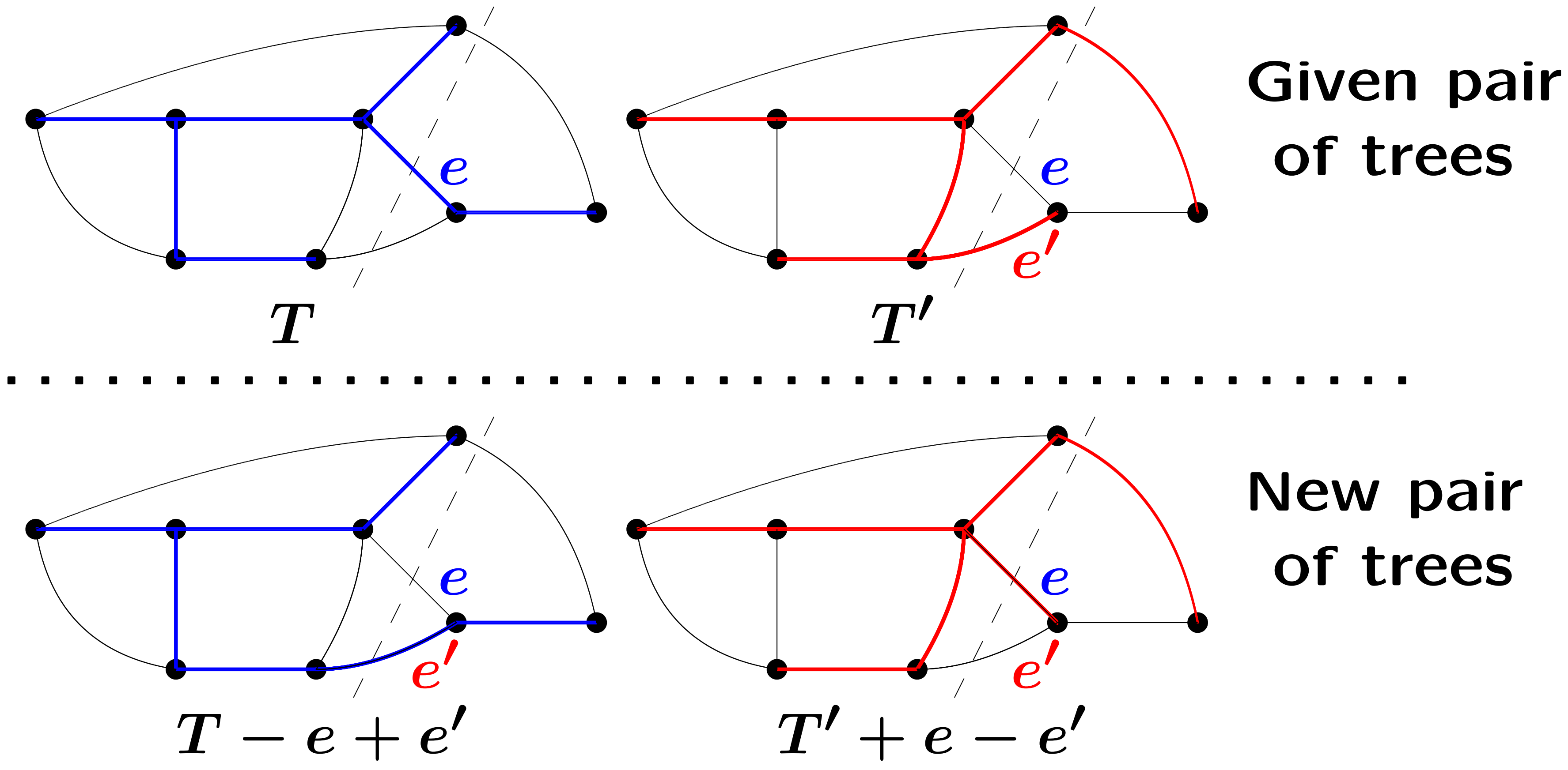
$$f(x) + f(y) \geq f(x - \alpha(x-y)) + f(y + \alpha(x-y))$$

$\Longrightarrow$ **Exchange** ($f : \mathrm{Z}^n \to \mathrm{R}$) $\quad \forall x, y, \quad \forall i : x_i > y_i$

$$f(x) + f(y) \geq \min \left[f(x - e_i) + f(y + e_i), \min_{x_j < y_j} \{ f(x - e_i + e_j) + f(y + e_i - e_j) \} \right]$$

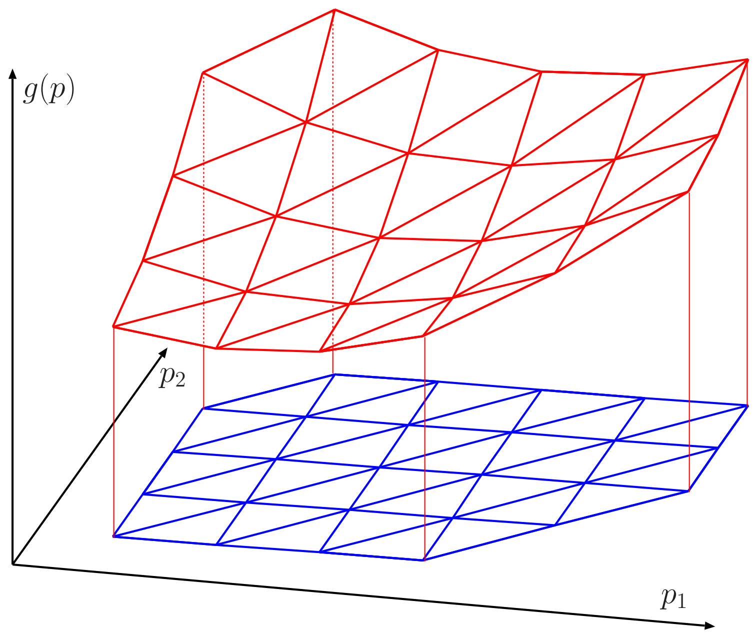
M♮-convex function (M = Matroid)

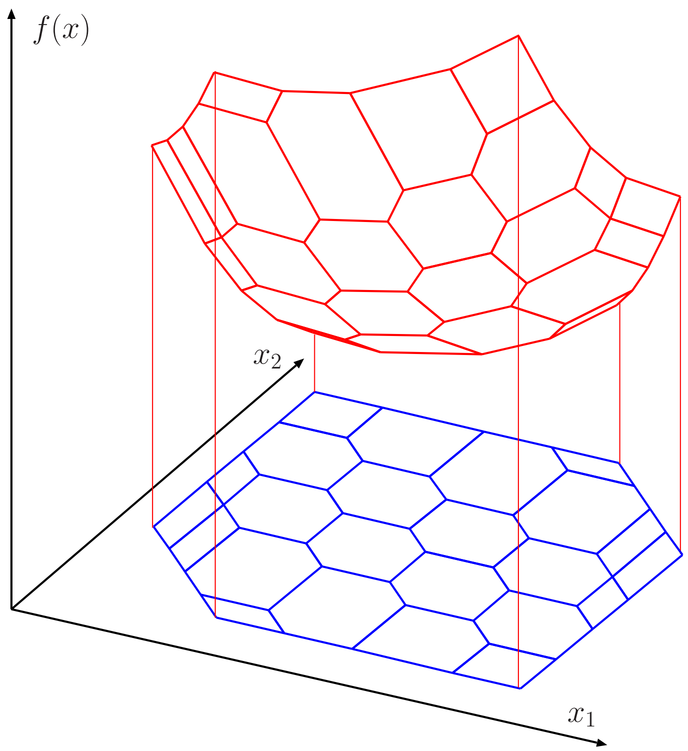
Tree: Exchange Property



Exchange property: For any $T, T' \in \mathcal{T}$, $e \in T \setminus T'$ there exists $e' \in T' \setminus T$ s.t. $T - e + e' \in \mathcal{T}$, $T' + e - e' \in \mathcal{T}$

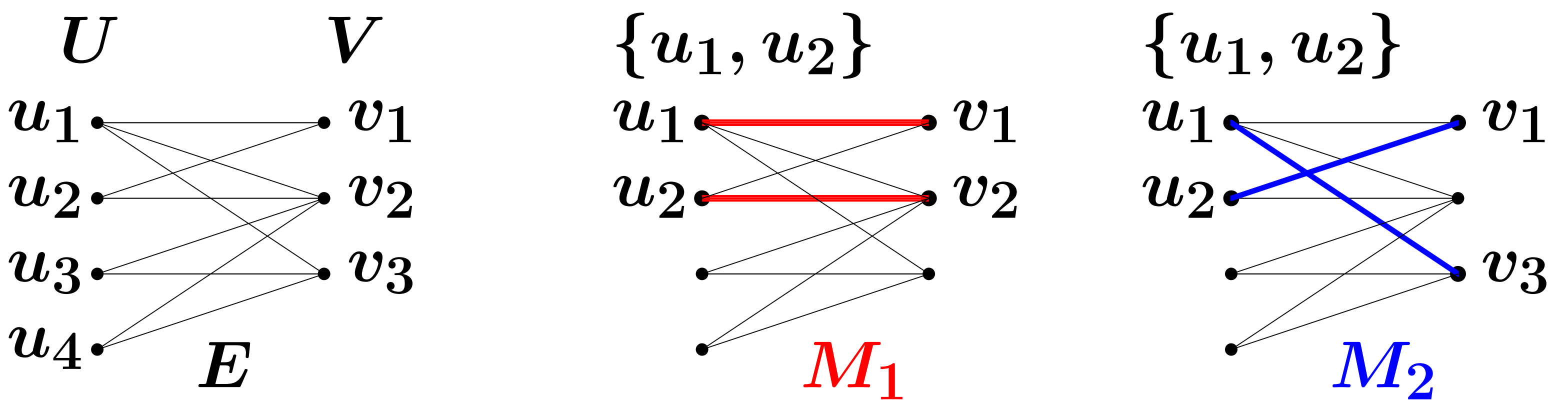
Discrete Convex Functions





L$^\natural$-convex fn **M$^\natural$-convex fn**

Matching / Assignment



Max weight for $X \subseteq U$ (w: given weight)

$$f(X) = \max\{\sum_{e \in M} w(e) \mid M\text{: matching, } U \cap \partial M = X\}$$

Max-weight function f is M$^\natural$-concave (Murota 96)

- Proof by augmenting path
- Extension to min-cost network flow

M$\natural$-concavity = Gross Substitutes

Think of f as a utility function

M$\natural$-concave $\iff$ Gross substitutes

Reijnierse–van Gallekom–Potters 02, Fujishige-Yang 03

Danilov-Koshevoy-Lang 03, M.-Tamura 03

Gross substitutes: **(f: utility, p: price)**

$x \in \arg\max(f - p)$, $p \leq q$,

$\Rightarrow \quad \exists\, y \in \arg\max(f - q)$: $y_i \geq x_i \quad$ if $p_i = q_i$

$\Longrightarrow$ **Applications to economics / game theory**

Polynomial Matrix

(Dress-Wenzel 90)

$$A = \begin{array}{|c|c|c|c|} \hline s+1 & s & 1 & 0 \\ 1 & 1 & 1 & 1 \\ \hline \end{array} \qquad \omega(J) = \deg \det A[J]$$

$\mathcal{B} = \{J \mid J$ is a base of column vectors$\}$

Grassman-Plücker $\Rightarrow$ Exchange (M$^\natural$-concave)

For any $J, J' \in \mathcal{B}$, $i \in J \setminus J'$, there exists $j \in J' \setminus J$
s.t. $J - i + j \in \mathcal{B}$, $J' + i - j \in \mathcal{B}$,

$$\omega(J) + \omega(J') \leq \omega(J - i + j) + \omega(J' + i - j)$$

Ex. $J = \{1, 2\}$, $\quad J' = \{3, 4\}$, $\quad i = 1$

$\det A[\{1,2\}] = \det A[\{3,4\}] = 1$, $\quad \omega(J) = \omega(J') = 0$

Can take $j = 3$: $\quad J - i + j = \{3, 2\}$, $\; J' + i - j = \{1, 4\}$

$\omega(J - i + j) = 1$, $\omega(J' + i - j) = 1$

Summary

From Continuous to Discrete

Convex: $\lambda f(x) + (1-\lambda) f(y) \geq f(\lambda x + (1-\lambda) y)$

Mid-pt convex: $f(x) + f(y) \geq 2 f(\frac{x+y}{2})$

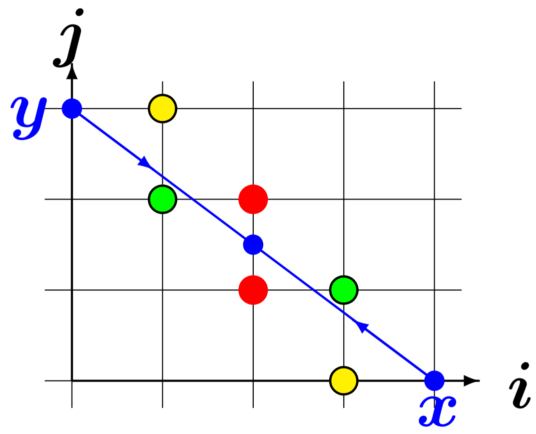
$\longrightarrow$ Discrete mid-pt conv: $\geq f\left(\left\lceil \frac{x+y}{2} \right\rceil\right) + f\left(\left\lfloor \frac{x+y}{2} \right\rfloor\right)$

Equi-dist: $f(x)+f(y) \geq f(x-\alpha(x-y))+f(y+\alpha(x-y))$

$\longrightarrow$ Exchange: $\geq \min \Big[f(x-e_i) + f(y+e_i),$

$\min_{x_j<y_j} \{ f(x-e_i+e_j) + f(y+e_i-e_j) \} \Big]$

Cont $\mathbb{R}^n \to \mathbb{R}$		Disc $\mathbb{Z}^n \to \mathbb{R}$
mid-pt conv	$\longrightarrow$	disc mid-pt
$\Updownarrow$	discr	(L$^\natural$-convex)
convex		
$\Updownarrow$	discr	(M$^\natural$-convex)
equi-dist conv	$\longrightarrow$	exchange



B3.

Conjugacy

— Legendre transform

Conjugacy in Polymatroids

Polyhedron S		**Submodular fn ρ**
$S = \{x \mid x(A) \le \rho(A) \ \forall A\}$	$\longleftarrow$	
	$\longrightarrow$	$\rho(A) = \max_{x \in S} x(A)$

Conjugacy in Polymatroids

Polyhedron S | **Submodular fn ρ**

$$S = \{x \mid x(A) \leq \rho(A) \ \forall A\} \ \longleftarrow$$

$$\longrightarrow \ \rho(A) = \max_{x \in S} x(A)$$

Indicator fn of S | **Lovász ext. of ρ**

$f(x) \in \{0, +\infty\}$ | $g(p)$

$$\rightarrow: \ g(p) = \max_{x \in S} \langle p, x \rangle = \max_{x} [\langle p, x \rangle - f(x)] = f^{\bullet}(p)$$

$$\leftarrow: \ f(x) = \max_{p} [\langle p, x \rangle - g(p)] = g^{\bullet}(x)$$

Legendre transform

M$^{\natural}$-convex | L$^{\natural}$-convex

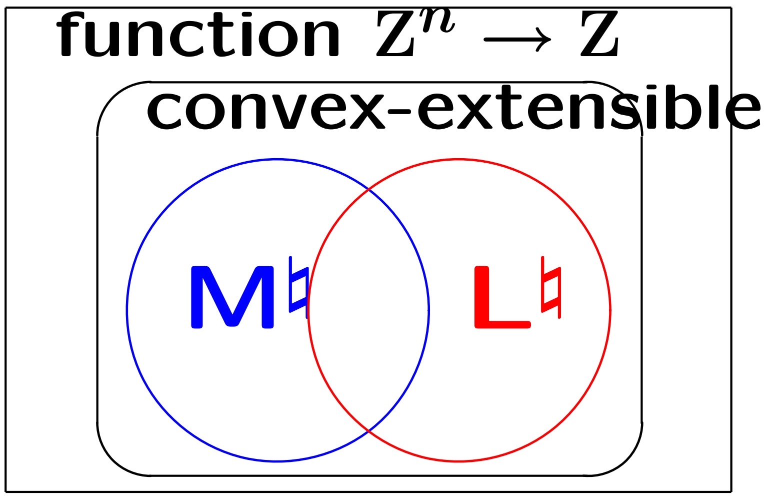
M-L Conjugacy Theorem

Integer-valued discrete fn $f : \mathbb{Z}^n \to \overline{\mathbb{Z}}$

Legendre transform: $f^\bullet(p) = \sup_{x \in \mathbb{Z}^n} [\langle p, x \rangle - f(x)]$

M♮-convex and L♮-convex are conjugate

$f \mapsto f^\bullet = g \mapsto g^\bullet = f$ **(Murota 98)**



History of Discrete Conjugacy

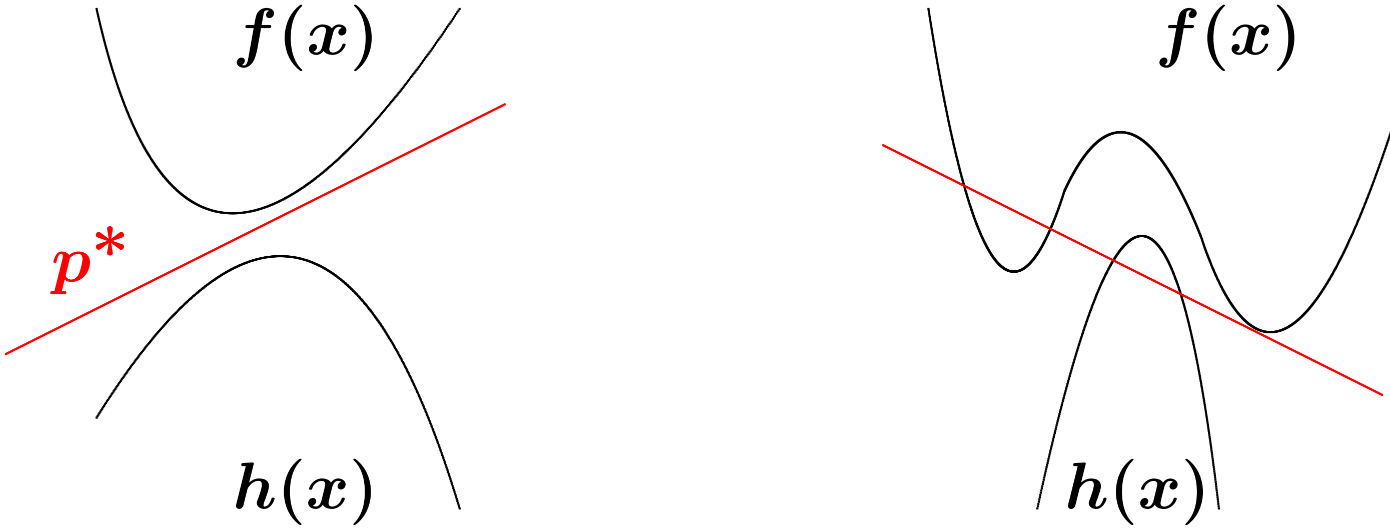
Matroid bases	$\longleftrightarrow$	Matroid rank fn
Whitney 35		Whitney 35
$\Downarrow$		$\Downarrow$
Polymatroid	$\longleftrightarrow$	**Submodular fn**
Edmonds 70		Edmonds 70
$\Downarrow$		$\Downarrow$
Valuated matroid	$\mid$	Lovász extension
Dress–Wenzel 90	$\mid$	Lovász 83
$\Downarrow$	$\mid$	$\Downarrow$
	$\mid$	Submod. integ. conv. fn
	$\mid$	Favati-Tardella 90
		$\Downarrow$
M-convex fn	$\longleftrightarrow$	**L-convex fn**
Murota 96		Murota 98
$\Updownarrow$		$\Updownarrow$
$M^\natural$-convex fn	$\longleftrightarrow$	$L^\natural$-convex fn
Murota-Shioura 99		Fujishige–Murota 00

B4.

Duality

Duality: Separation Theorem

main issue in convexity paradigm



$f : \mathrm{R}^n \to \mathrm{R}$ **convex**

$h : \mathrm{R}^n \to \mathrm{R}$ **concave**

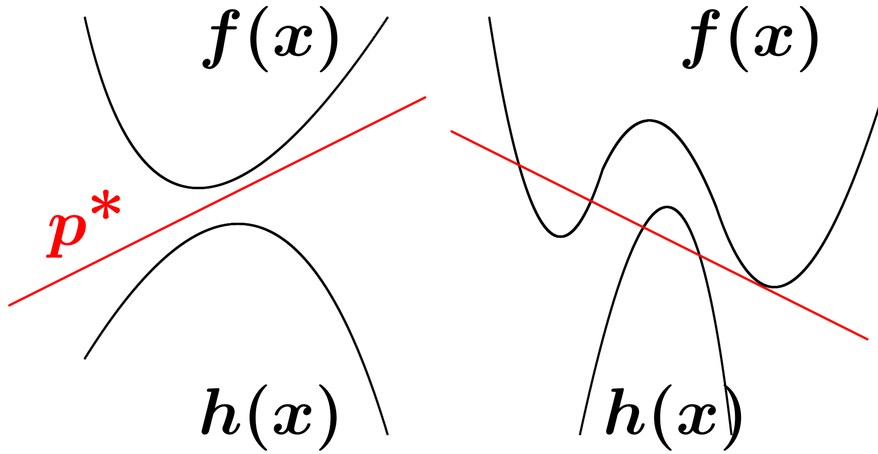
$$f(x) \geq h(x) \quad (\forall x) \Rightarrow \exists\ \alpha^* \in \mathrm{R}, \quad \exists\ p^* \in \mathrm{R}^n:$$

$$f(x) \geq \alpha^* + \langle p^*, x \rangle \geq h(x) \qquad (x \in \mathrm{R}^n)$$

Discrete Separation Theorem

$f : \mathbb{Z}^n \to \mathbb{R}$ **"convex"**

$h : \mathbb{Z}^n \to \mathbb{R}$ **"concave"**



- $f(x) \geq h(x) \quad (\forall x \in \mathbb{Z}^n) \Rightarrow \quad \exists\ \alpha^* \in \mathbb{R}, \quad \exists\ p^* \in \mathbb{R}^n$:

$$f(x) \geq \alpha^* + \langle p^*, x \rangle \geq h(x) \qquad (x \in \mathbb{Z}^n)$$

- f, h: **integer-valued** $\Rightarrow \alpha^* \in \mathbb{Z}, \quad p^* \in \mathbb{Z}^n$

Frank's Discrete Separation

(Frank 82)

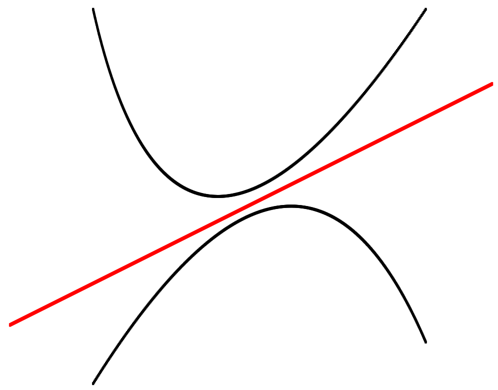
$\rho : 2^V \to \mathrm{R}$: **submodular** $\quad (\rho(\emptyset) = 0)$

$\mu : 2^V \to \mathrm{R}$: **supermodular** $\quad (\mu(\emptyset) = 0)$

- $\rho(X) \geq \mu(X) \quad (\forall X \subseteq V) \Rightarrow \exists\ x^* \in \mathrm{R}^V$:

$$\rho(X) \geq x^*(X) \geq \mu(X) \qquad (\forall X \subseteq V)$$

- ρ, μ: **integer-valued** $\Rightarrow x^* \in \mathrm{Z}^V$

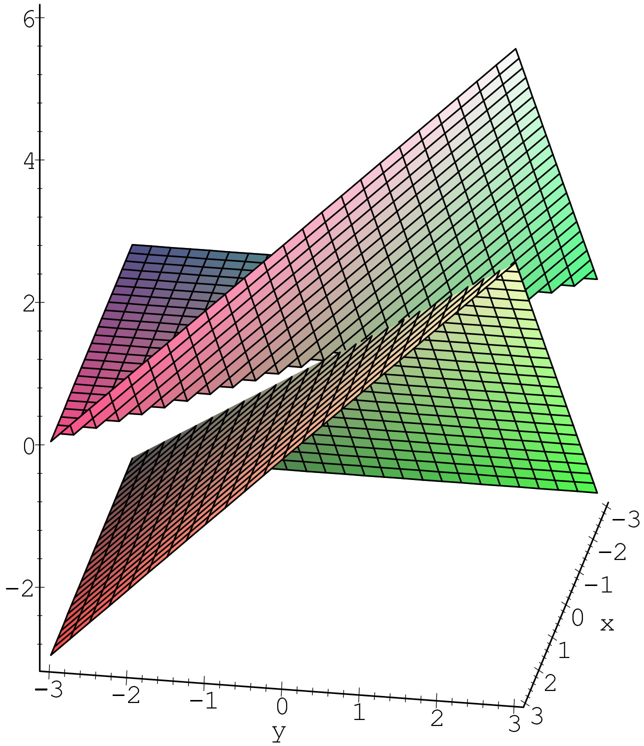


Equivalent to Edmonds' polymatroid intersection

Difficulty of Discrete Separation (1)

$f(x, y) = \max(0, x + y)$ **convex**

$h(x, y) = \min(x, y)$ **concave**



separable but nonintegral

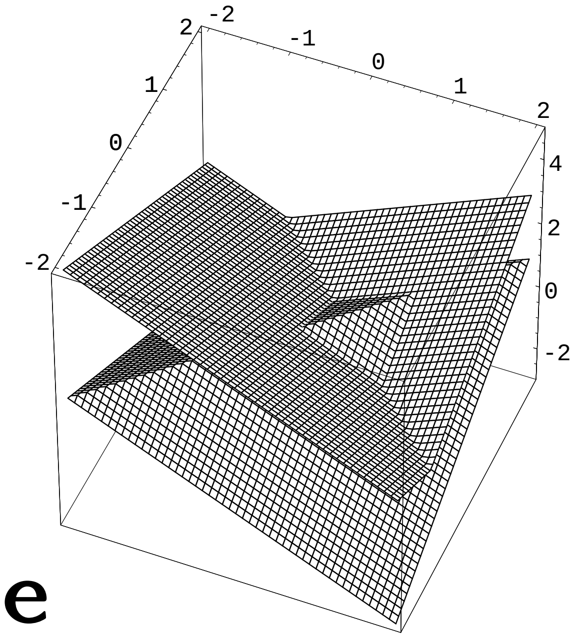
$p^* = (1/2, 1/2)$, $\alpha^* = 0$ **unique separating plane**

Difficulty of Discrete Separation (2)

Even real-separation is nontrivial

$f(x,y) = |x+y-1|$ convex

$h(x,y) = 1 - |x-y|$ concave



- $f(x,y) \geq h(x,y)$ $(\forall (x,y) \in \mathbf{Z}^2)$ true
- $f(x,y) \geq h(x,y)$ $(\forall (x,y) \in \mathbf{R}^2)$ not true

since $f = 0 < h = 1$ at $(x,y) = (1/2, 1/2)$

$\Longrightarrow$ No $\alpha^* \in \mathbf{R}$, $p^* \in \mathbf{R}^2$ satisfies

$$f(x) \geq \alpha^* + \langle p^*, x \rangle \geq h(x)$$

Discrete Separation Theorems

(Murota 96/98)

M-separation Thm: $\mathrm{M}^\natural$-convex fn

$\supset$ Weight splitting for weighted matroid intersection

(Iri-Tomizawa 76, Frank 81)

(linear fn, indicator fn $=$ $\mathrm{M}^\natural$-convex fn)

L-separation Thm: $\mathrm{L}^\natural$-convex fn

$\supset$ Discrete separation for submod. set function

(Frank 82)

(submod. set fn $=$ $\mathrm{L}^\natural$-convex fn on 0–1 vectors)

M-convex Intersection: Min [M$^\natural$+M$^\natural$]

M$^\natural$+M$^\natural$ is NOT M$^\natural$

f_1, f_2 : M$^\natural$-convex $(\mathbf{Z}^n \to \mathbf{R})$, $x^* \in \mathrm{dom} f_1 \cap \mathrm{dom} f_2$

(1) x^* minimizes $\boxed{f_1 + f_2}$ (Murota 96)

$\iff$ $\exists\, p$ (certificate of optimality)

- x^* minimizes $\boxed{f_1(x) - \langle p, x \rangle}$ (M-opt thm)
- x^* minimizes $\boxed{f_2(x) + \langle p, x \rangle}$ (M-opt thm)

(2) $\mathrm{argmin}\,(f_1 + f_2) = \mathrm{argmin}\,(f_1 - p) \cap \mathrm{argmin}\,(f_2 + p)$

(3) f_1, f_2 are integer-valued $\Rightarrow$ integral p

$\Longrightarrow$ Frank's weight splitting thm for matroid intersec.

Min-Max Duality

Legendre–Fenchel transform

$$f^{\bullet}(p) = \sup\{\langle p, x\rangle - f(x) \mid x \in \mathbf{Z}^n\}$$

$$h^{\circ}(p) = \inf\{\langle p, x\rangle - h(x) \mid x \in \mathbf{Z}^n\}$$

Fenchel-type duality thm (Murota 96, 98)

f: M$^{\natural}$-convex $\quad$ h: M$^{\natural}$-concave $\quad$ $(\mathbf{Z}^n \to \mathbf{Z})$

$$\inf_{x \in \mathbf{Z}^n} \{f(x) - h(x)\} = \sup_{p \in \mathbf{Z}^n} \{h^{\circ}(p) - f^{\bullet}(p)\}$$

self-conjugate $\quad$ ($f^{\bullet}$: L$^{\natural}$-conv $\quad$ h°: L$^{\natural}$-conv)

$\Longrightarrow$ Edmonds' matroid intersection thm

Relation among Duality Thms

Discrete Convex	Combinatorial Opt.
M-separation	
$f(x) \geq \boxed{\text{Lin}} \geq h(x)$	Fenchel duality (Fujishige 84)
	matroid intersect. (Edmonds 70)
$\Updownarrow$	$\Updownarrow$
Fenchel duality	$\Rightarrow$ discrete separ. for submod (Frank 82)
$\inf\{f - h\} = \sup\{h^\circ - f^\bullet\}$	$\Rightarrow$ valuated matroid intersect. (M. 96)
$\Updownarrow$	$\Downarrow$
L-separation	weighted matroid intersect.
$f^\bullet(p) \geq \boxed{\text{Lin}} \geq h^\circ(p)$	(Edmonds 79, Iri-Tomizawa 76, Frank 81)

B5.

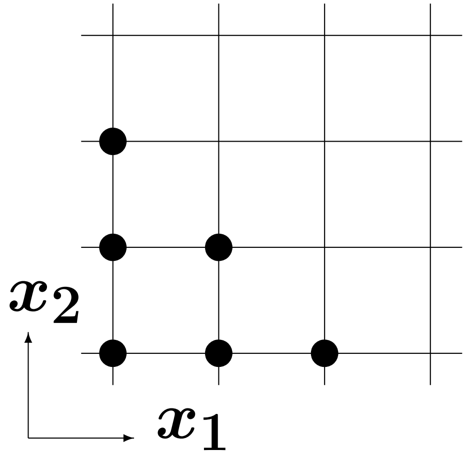
Discrete Hessian

Discrete Hessian Matrix

$$H_{ij}(x) = f(x + e_i + e_j) - f(x + e_i) - f(x + e_j) + f(x)$$

$$H_{11}(0,0) = f(2,0) - 2f(1,0) + f(0,0)$$

$$H_{12}(0,0) = f(1,1) - f(1,0) - f(0,1) + f(0,0)$$



Question:

Hessian $\succeq O$ $\Leftarrow ? \Rightarrow$ Convex Extensible

$\not\Leftarrow$ by a simple example

$$f(x_1, x_2) = |x_1 - x_2| \qquad H(0,0) = \begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix}$$

Hessian $\succeq O \not\Rightarrow$ Convex Extensible

Counterexample by semidefinite programming

(Moriguchi–M. 12)

$f : \mathbf{Z}^2 \to \mathbf{Z}$

$H(x_1, x_2) \succeq O$

x_2 \ x_1	0	1	2	3	4	5	6	7	8	9	10
10	100										
9	73	45									
8	50	21	−4								
7	29	1	−25	−47							
6	11	−17	−43	−64	−56						
5	−3	−32	−57	−79	−71	−59					
4	−15	−44	−69	−91	−82	−71	−56				
3	−24	−52	−78	**−100**	−91	−79	−64	−47			
2	−23	−48	**−66**	−78	−69	−57	−43	−25	−4		
1	−15	**−35**	−48	−52	−44	−32	−17	1	21	45	
0	1	−15	−23	−24	−15	−3	11	29	50	73	100

convexity fails ($-35 - 2 \times (-66) + (-100) = -3$)

Hessian for M♮-/L♮-convexity

M♮-convexity: (Hirai-M. 04, M. 07)

$f : \mathbb{Z}^n \to \mathbb{R}$ is M♮-convex $\iff$ For each $x \in \mathbb{Z}^n$:

- $H_{ij}(x) \geq \min(H_{ik}(x), H_{jk}(x))$ if $\{i,j\} \cap \{k\} = \emptyset$
- $H_{ij}(x) \geq 0$ for any (i,j)

combinatorial cond. stronger than $H(x) \succeq O$

cf. "ultra metric" in finite metric space

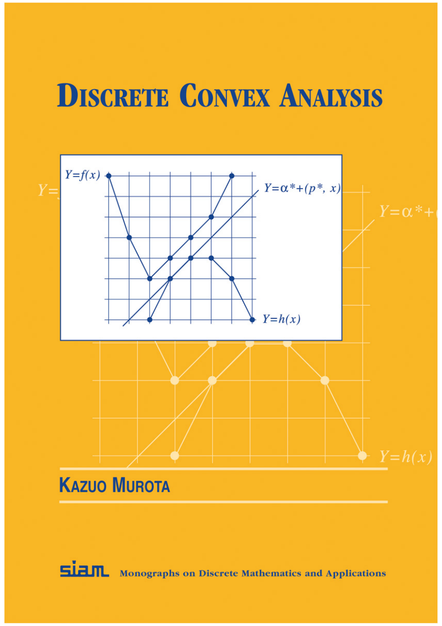
L♮-convexity: modified $\tilde{H}_{ij}(x)$ (Moriguchi-M. 05)

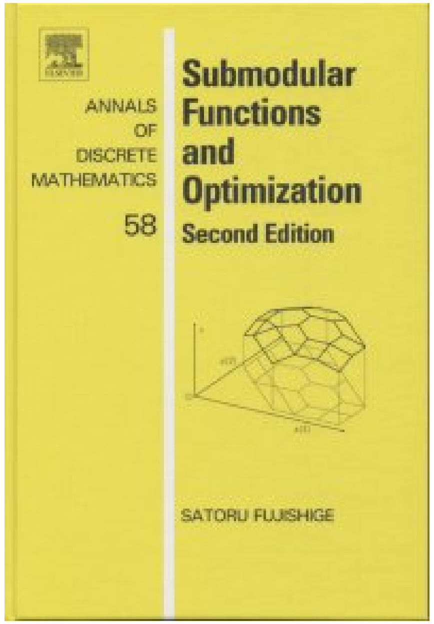
- $\tilde{H}_{ii}(x) \geq \Sigma_{j \neq i} |\tilde{H}_{ij}(x)|$
- $\tilde{H}_{ij}(x) \leq 0$ for any $i \neq j$

Books/Surveys

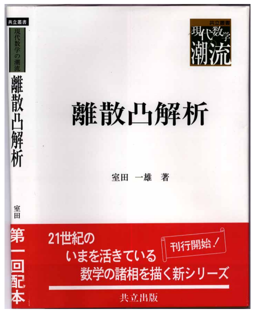
Murota: Discrete Convex Analysis, SIAM, 2003

Fujishige: Submodular Functions and Optimization, 2nd ed., Elsevier, 2005 (Chap. VII)

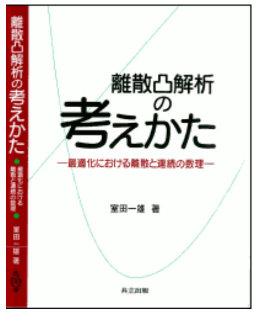




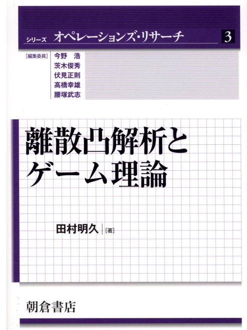
M. 01



M. 07



Tamura 09



Murota: Recent developments in discrete convex analysis, in: Research Trends in Combinatorial Optimization, Bonn 2008, Springer, 2009, Chap. 11, 219–260.

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