RIMS, August 30, 2012

Introduction to
Discrete Convex Analysis

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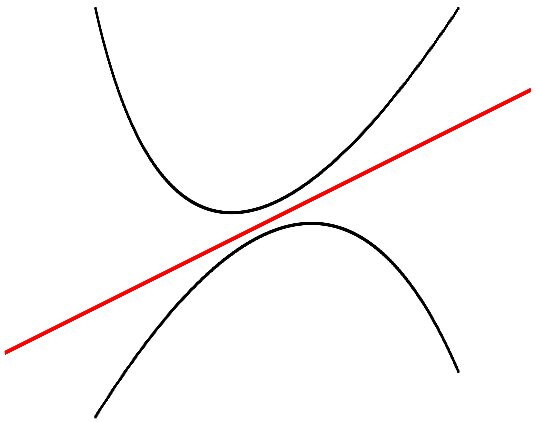

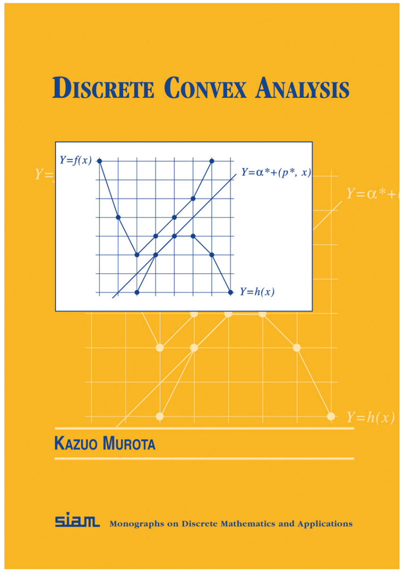

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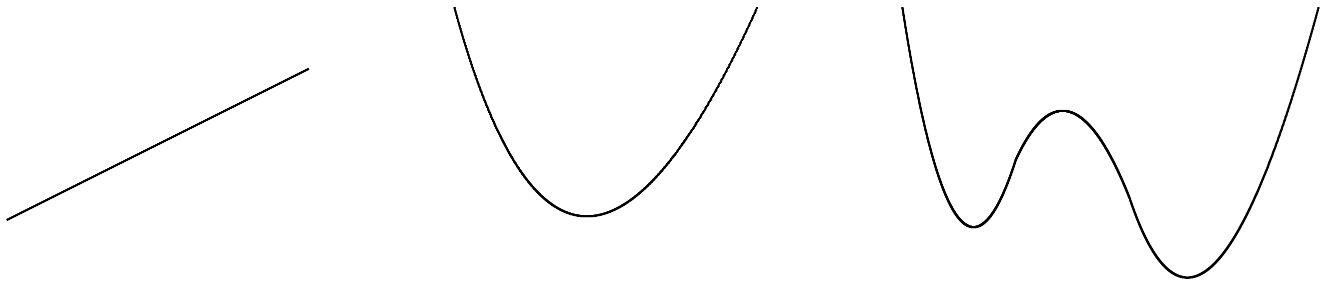
Convexity Paradigm

NOT Linear vs N o n l i n e a r



BUT C o n v e x vs Nonconvex

Convexity in discrete optimization?

$\Longrightarrow$ Convexity + Combinatorics

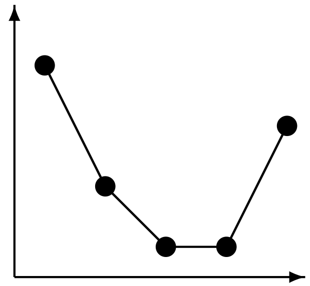
$$f : \mathbf{Z}^n \to \mathbf{Z}, \quad \mathbf{Z}^n \to \mathbf{R}, \quad \mathbf{R}^n \to \mathbf{R}$$

Discrete Convex Analysis

Convexity Paradigm in Discrete Optimization

Matroid Theory + Convex Analysis

Submodular fn	$\leftrightarrow$	Matroid base
$\Downarrow$		$\Downarrow$
L-convex fn	$\leftrightarrow$	M-convex fn



- Global optimality $\Longleftrightarrow$ local optimality
- Conjugacy: Legendre–Fenchel transform
- Duality (Fenchel min-max, discrete separation)
- Minimization algorithms
- Applications: OR, game, economics, matrices

Some History

1935	Matroid	Whitney
1965	Submodular function	Edmonds
1975	Engrg application of matroid	Iri, Recski
1983	Submodularity and convexity	Lovász, Frank, Fujishige
1990	Valuated matroid	Dress–Wenzel
	Integrally convex fn	Favati–Tardella
1996	Discrete convex analysis	Murota
2000	Submod. fn minimization algorithm	Iwata–Fleischer–Fujishige, Schrijver

Contents

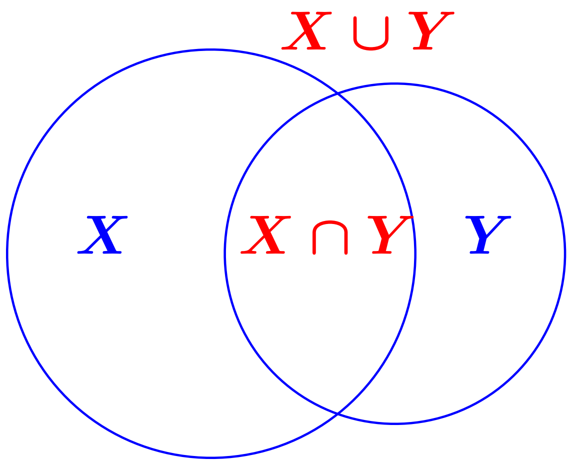
B1.

Submodularity and Convexity (1980's)

Submodular Function

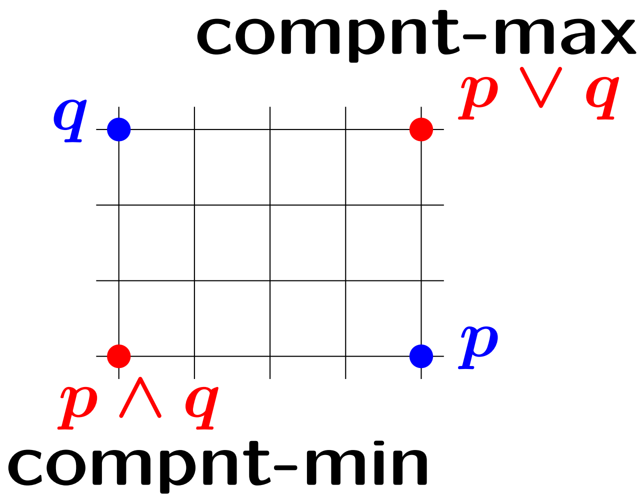
Set function ρ is submodular:

$$\rho(X) + \rho(Y) \geq \rho(X \cup Y) + \rho(X \cap Y)$$



$g : \mathbb{Z}^n \to \mathbb{R}$ is submodular:

$$g(p) + g(q) \geq g(p \vee q) + g(p \wedge q)$$



Submodularity & Convexity in 1980's

$$\rho(X) + \rho(Y) \geq \rho(X \cup Y) + \rho(X \cap Y)$$

- min/max algorithms (Grötschel–Lovász–Schrijver/ Jensen–Korte, Lovász)

  min $\Rightarrow$ polynomial, max $\Rightarrow$ NP-hard

- Convex extension (Lovász)

  set fn is submod $\Leftrightarrow$ Lovász ext is convex

- Duality theorems (Edmonds, Frank, Fujishige)

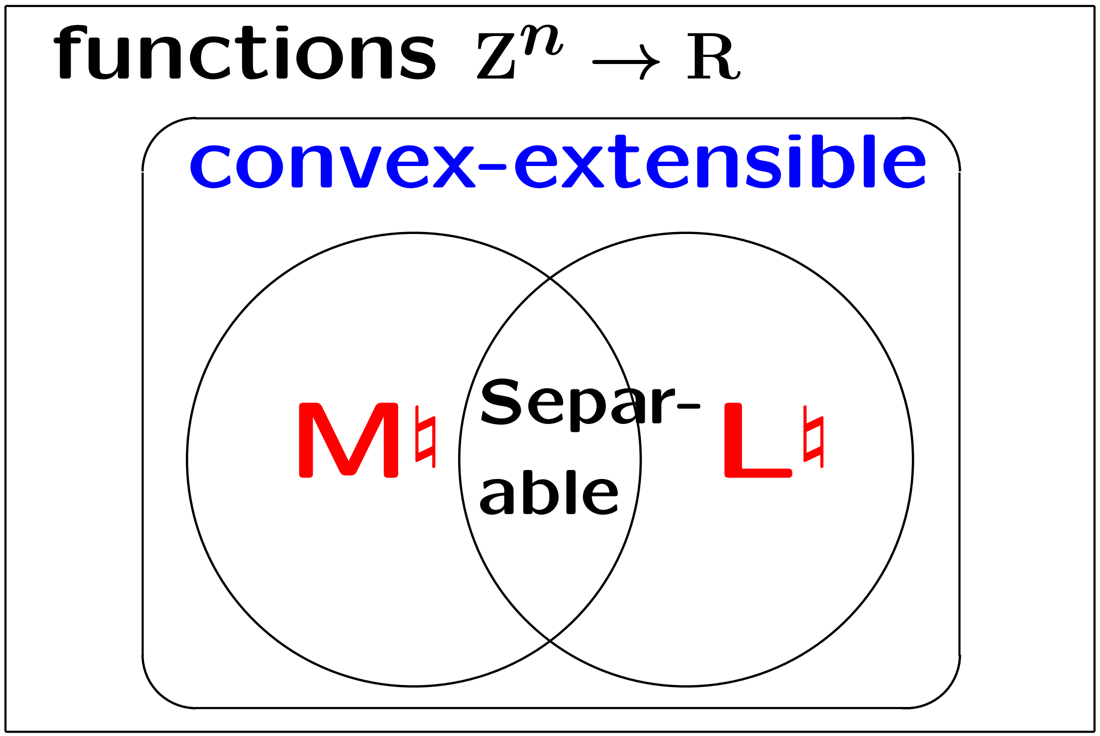
  discrete separation, Fenchel min-max

Duality for submodular set functions = Convexity + Discreteness

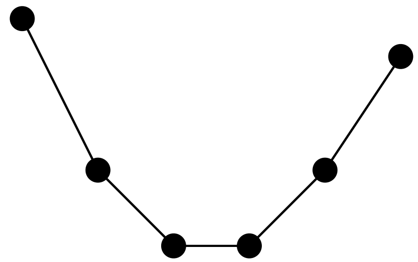
B2.

L-convex and M-convex Functions

Discrete Convex Functions



$$f : \mathbf{Z}^n \to \mathbf{R}$$



f is convex-extensible

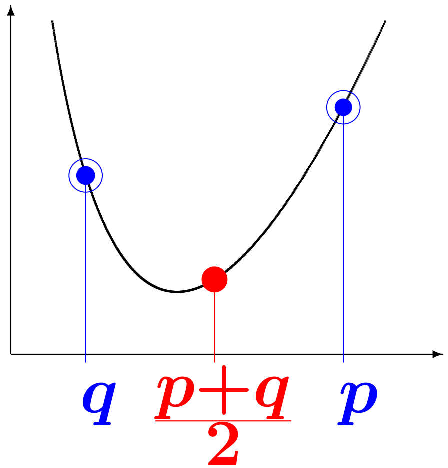
$\Leftrightarrow$ $\exists$ convex $\overline{f}$:

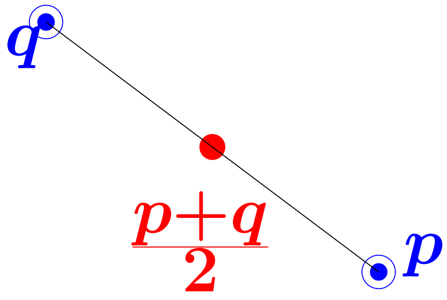
$$f(x) = \overline{f}(x)$$

Convex-extensibility does not help much

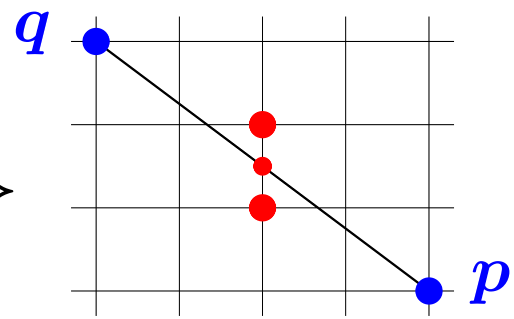
L♮-convexity from Mid-pt-convexity

(Murota 98, Fujishige–Murota 00)





$\Longrightarrow$



Mid-point convex ($g : \mathbf{R}^n \to \mathbf{R}$):

$$g(p) + g(q) \geq 2g\left(\frac{p+q}{2}\right)$$

$\Longrightarrow$ **Discrete mid-point convex ($g : \mathbf{Z}^n \to \mathbf{R}$)**

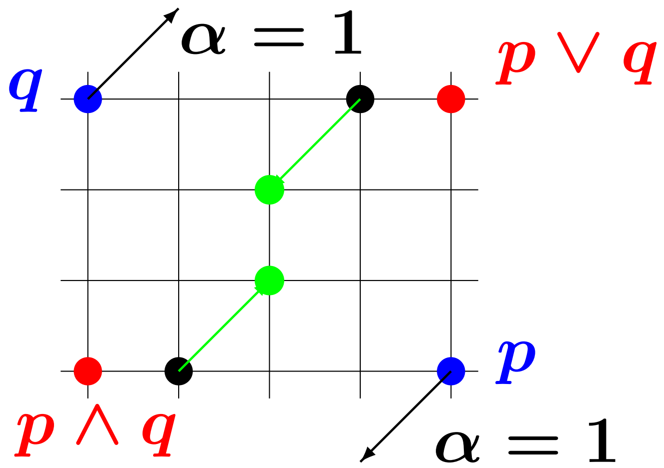
$$g(p) + g(q) \geq g\left(\left\lceil \frac{p+q}{2} \right\rceil\right) + g\left(\left\lfloor \frac{p+q}{2} \right\rfloor\right)$$

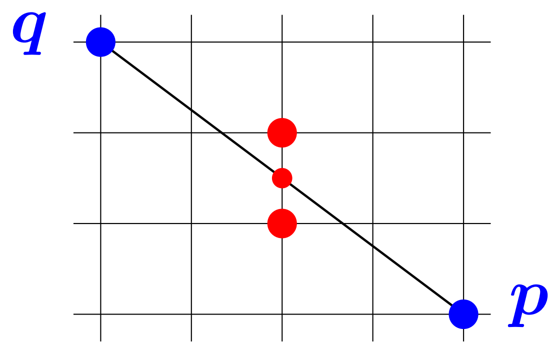
L♮-convex function (L = Lattice)

Translation Submodularity (L♮)

$$g(p) + g(q) \geq g((p - \alpha 1) \vee q) + g(p \wedge (q + \alpha 1))$$

$(\alpha \geq 0)$





discrete mid-pt convex

$\tilde{g}(p_0, p) = g(p - p_0 1)$ is submodular in (p_0, p)

$\Leftrightarrow$ translation submodular (Fujishige-Murota 00)

$\Leftrightarrow$ discrete mid-pt convex (Fujishige-Murota 00)

$\Leftrightarrow$ submod. integ. convex (Favati-Tardella 90)

L♮-convex Function: Examples

Quadratic: $g(p) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} p_i p_j$ is L♮-convex

$$\Leftrightarrow a_{ij} \leq 0 \quad (i \neq j), \qquad \sum_{j=1}^{n} a_{ij} \geq 0 \ (\forall i)$$

Separable convex: For univariate convex ψ_i and ψ_{ij}

$$g(p) = \sum_{i=1}^{n} \psi_i(p_i) + \sum_{i \neq j} \psi_{ij}(p_i - p_j)$$

Range: $g(p) = \max\{p_1, p_2, \ldots, p_n\} - \min\{p_1, p_2, \ldots, p_n\}$

Submodular set function: $\rho : 2^V \to \overline{\mathrm{R}}$

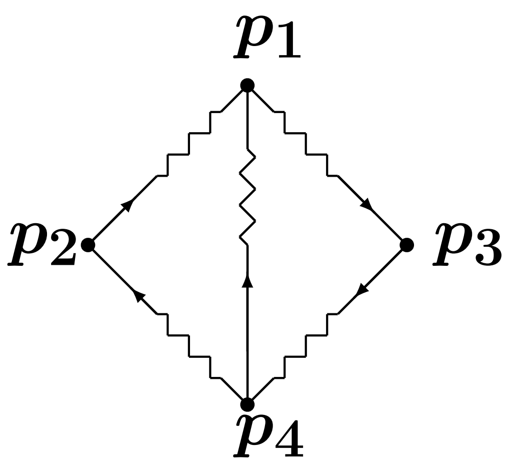
$\Leftrightarrow \rho(X) = g(\chi_X)$ for L♮-convex g

Multimodular: $h : \mathrm{Z}^n \to \overline{\mathrm{R}}$ is multimodular $\Leftrightarrow$

$h(p) = g(p_1, p_1 + p_2, \ldots, p_1 + \cdots + p_n)$ for L♮-convex g

Node-branch incidende (graph structure)

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 \end{bmatrix}$$



Node admittance

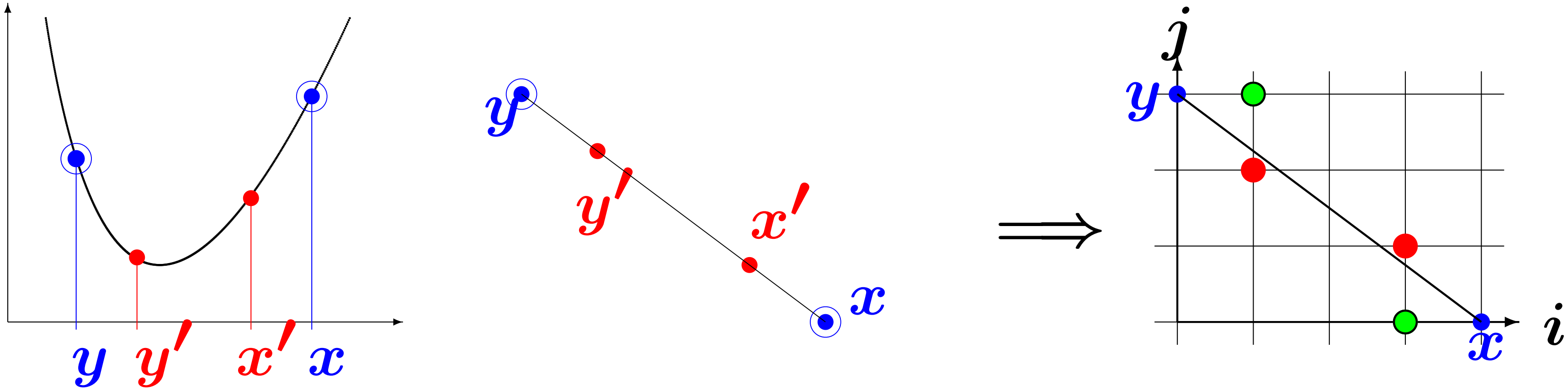
$$L = A \ \mathrm{diag}\,(g_j) \ A^\top =$$

$$\begin{bmatrix} g_1+g_2+g_5 & -g_1 & -g_2 & -g_5 \\ -g_1 & g_1+g_4 & 0 & -g_4 \\ -g_2 & 0 & g_2+g_3 & -g_3 \\ -g_5 & -g_4 & -g_3 & g_3+g_4+g_5 \end{bmatrix}$$

(off-diag ≤ 0, row sum $= 0$)

M♮-convexity from Equi-dist-convexity

(Murota 96, Murota–Shioura 99)



Equi-distance convex ($f : \mathbf{R}^n \to \mathbf{R}$):

$$f(x) + f(y) \geq f(x - \alpha(x - y)) + f(y + \alpha(x - y))$$

$\Longrightarrow$ **Exchange** ($f : \mathbf{Z}^n \to \mathbf{R}$) $\quad \forall x, y, \quad \forall i : x_i > y_i$

$$f(x) + f(y) \geq \min \left[f(x - e_i) + f(y + e_i), \right.$$

$$\left. \min_{x_j < y_j} \{ f(x - e_i + e_j) + f(y + e_i - e_j) \} \right]$$

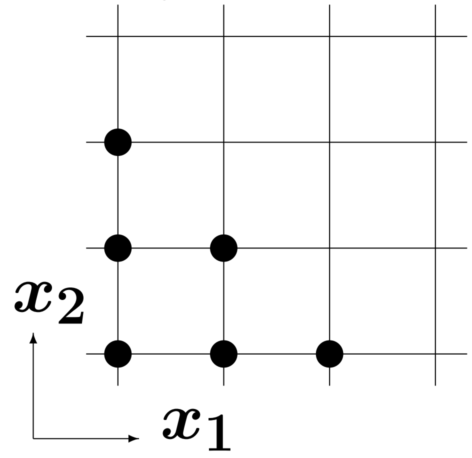
M♮-convex function $\qquad$ **(M = Matroid)**

Discrete Hessian for M♮-convexity

$$H_{ij}(x) = f(x + e_i + e_j) - f(x + e_i) - f(x + e_j) + f(x)$$

$H_{11}(0,0) = f(2,0) - 2f(1,0) + f(0,0)$

$H_{12}(0,0) = f(1,1) - f(1,0) - f(0,1) + f(0,0)$



M♮-convexity: (Hirai-M. 04, M. 07)

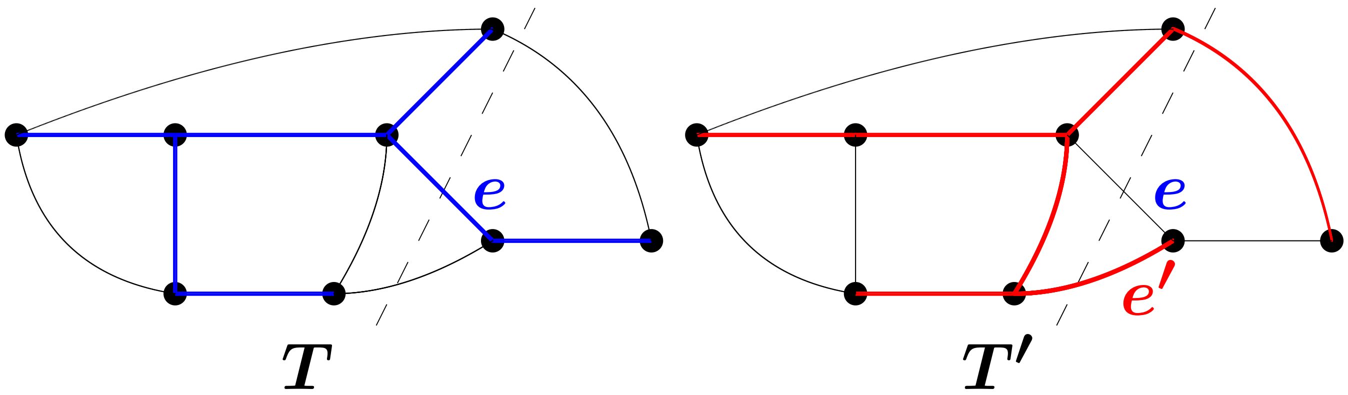
$f : \mathbb{Z}^n \to \mathbb{R}$ is M♮-convex

$\Longleftrightarrow$ For each $x \in \mathbb{Z}^n$:

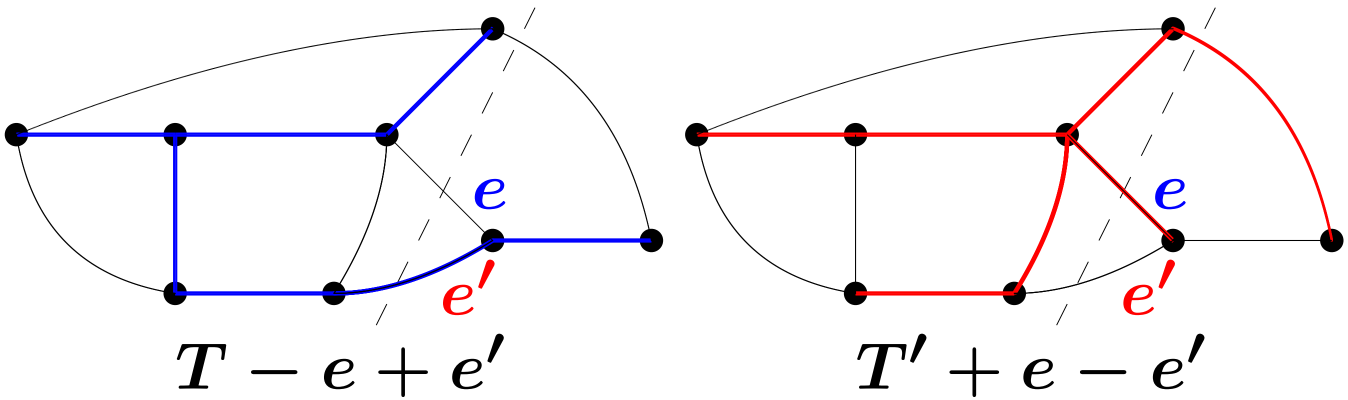
- $H_{ij}(x) \geq \min(H_{ik}(x), H_{jk}(x))$ if $\{i,j\} \cap \{k\} = \emptyset$
- $H_{ij}(x) \geq 0$ for any (i,j)

cf. "ultra metric" in finite metric space

Tree: Exchange Property



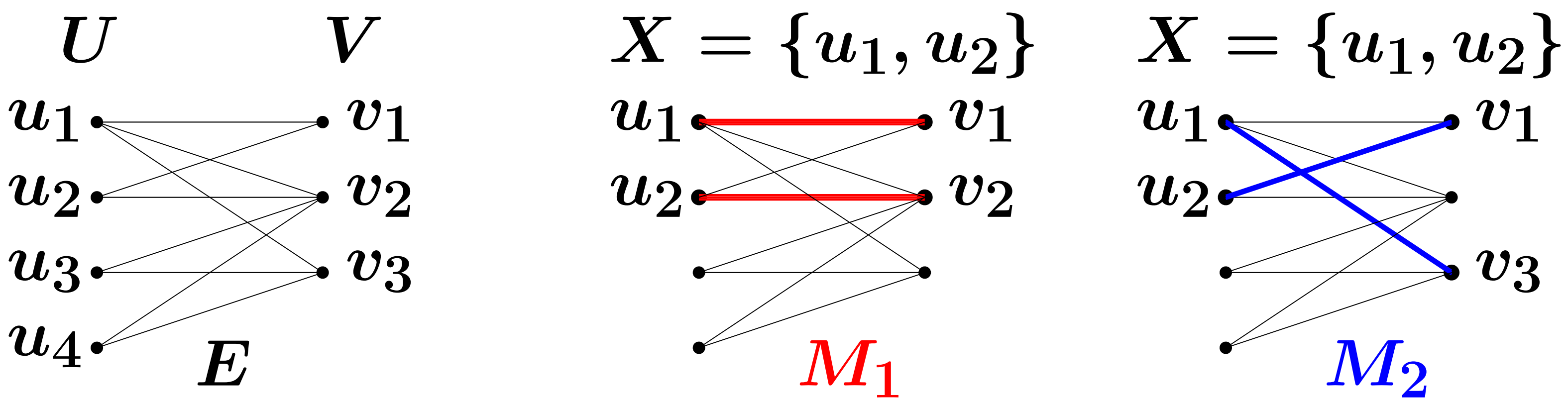
Given pair of trees



New pair of trees

Exchange property: For any $T, T' \in \mathcal{T}$, $e \in T \setminus T'$ there exists $e' \in T' \setminus T$ s.t. $T - e + e' \in \mathcal{T}$, $T' + e - e' \in \mathcal{T}$

Matching / Assignment



Max weight for $X \subseteq U$ (w: given weight)

$$f(X) = \max\{\sum_{e \in M} w(e) \mid M\text{: matching, } U \cap \partial M = X\}$$

Max-weight function f is M$^\natural$-concave (Murota 96)

- Proof by augmenting path
- Extension to min-cost network flow

Gross Substitutes

$f : 2^V \to \mathrm{R}$ utility (reservation value) function

p price vector

$D(p) = \arg\max(f - p) = \{X \mid f(X) - p(X)$ is maximum$\}$

demand correspondence

Gross substitutes property: (Kelso–Crawford 82)

$X \in D(p), \ \ p \leq q$

$\Rightarrow \ \ \exists\, Y \in D(q)$: $\{i \in X \mid p_i = q_i\} \ \subseteq \ Y$

Equiv. cond. for $D(p)$ (Gul–Stacchetti 99)

Equiv. cond. for f (Reijnierse–van Gallekom–Potters 02)

& equivalence to M$^\natural$-concavity (Fujishige–Yang 03)

$\Longrightarrow$ To be extended for $f : \mathrm{Z}^n \to \mathrm{R}$

Gross Substitutes for f (not for $D(p)$)

$f: 2^V \to \mathrm{R}$ (set function)

f: gross substitutes $\iff$

(i) $f(S \cup \{i,j\}) + f(S) \leq f(S \cup \{i\}) + f(S \cup \{j\})$

(submodular)

(ii) $f(S \cup \{i,j\}) + f(S \cup \{k\}) \leq$

$$\max[f(S \cup \{i,k\}) + f(S \cup \{j\}), f(S \cup \{j,k\}) + f(S \cup \{i\})]$$

(Reijnierse–van Gallekom–Potters 02)

cf. Local exchange axiom of M$^\natural$-concave functions

Polynomial Matrix (Dress-Wenzel 90)

$$A = \begin{array}{|c|c|c|c|} \hline s+1 & s & 1 & 0 \\ 1 & 1 & 1 & 1 \\ \hline \end{array} \qquad \omega(J) = \deg \det A[J]$$

$\mathcal{B} = \{J \mid J$ is a base of column vectors$\}$

Grassman-Plücker $\Rightarrow$ Exchange (M-concave)

For any $J, J' \in \mathcal{B}$, $i \in J \setminus J'$, there exists $j \in J' \setminus J$
s.t. $J - i + j \in \mathcal{B}$, $J' + i - j \in \mathcal{B}$,

$$\omega(J) + \omega(J') \leq \omega(J - i + j) + \omega(J' + i - j)$$

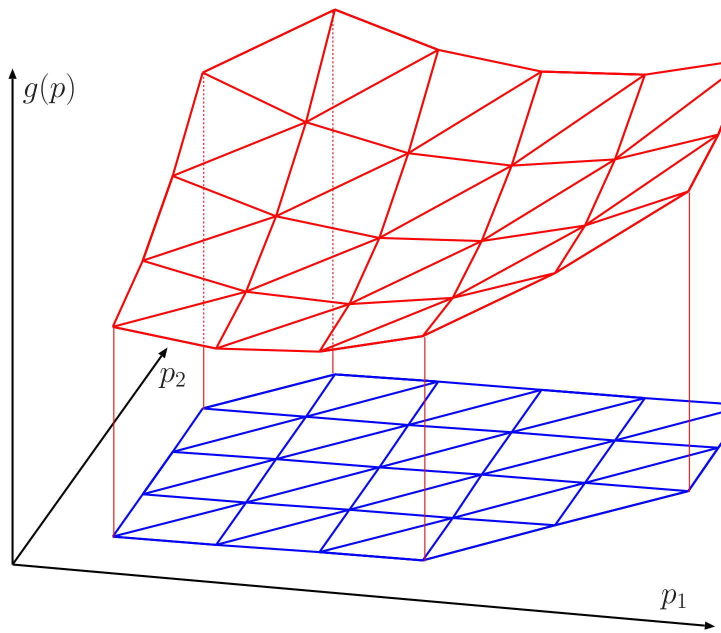
Ex. $J = \{1, 2\}$, $\quad J' = \{3, 4\}$, $\quad i = 1$

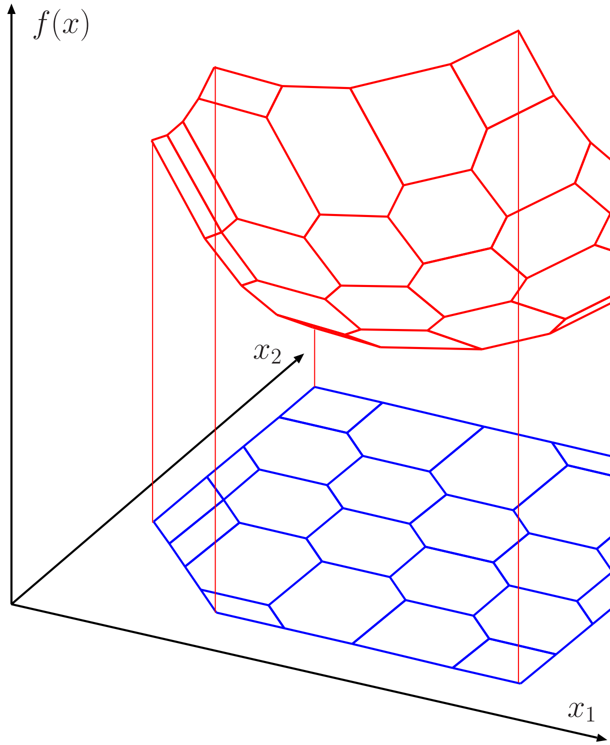
$\det A[\{1, 2\}] = \det A[\{3, 4\}] = 1$, $\quad \omega(J) = \omega(J') = 0$

Can take $j = 3$: $\quad J - i + j = \{3, 2\}$, $J' + i - j = \{1, 4\}$

$\omega(J - i + j) = 1$, $\omega(J' + i - j) = 1$

Discrete Convex Functions





L♮-convex fn **M♮-convex fn**

B3.

Conjugacy and Duality

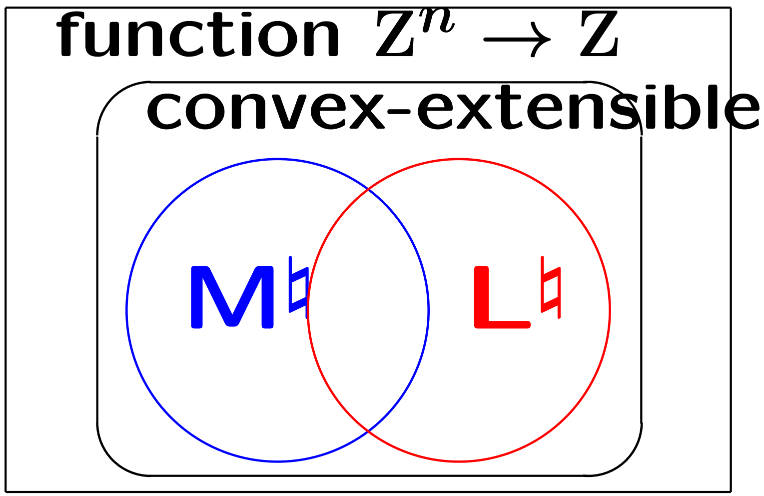
M-L Conjugacy Theorem

Integer-valued discrete fn $f : \mathbb{Z}^n \to \overline{\mathbb{Z}}$

Legendre transform: $$f^\bullet(p) = \sup_{x \in \mathbb{Z}^n} [\langle p, x \rangle - f(x)]$$

M♮-convex and L♮-convex are conjugate

$f \mapsto f^\bullet = g \mapsto g^\bullet = f$ **(Murota 98)**

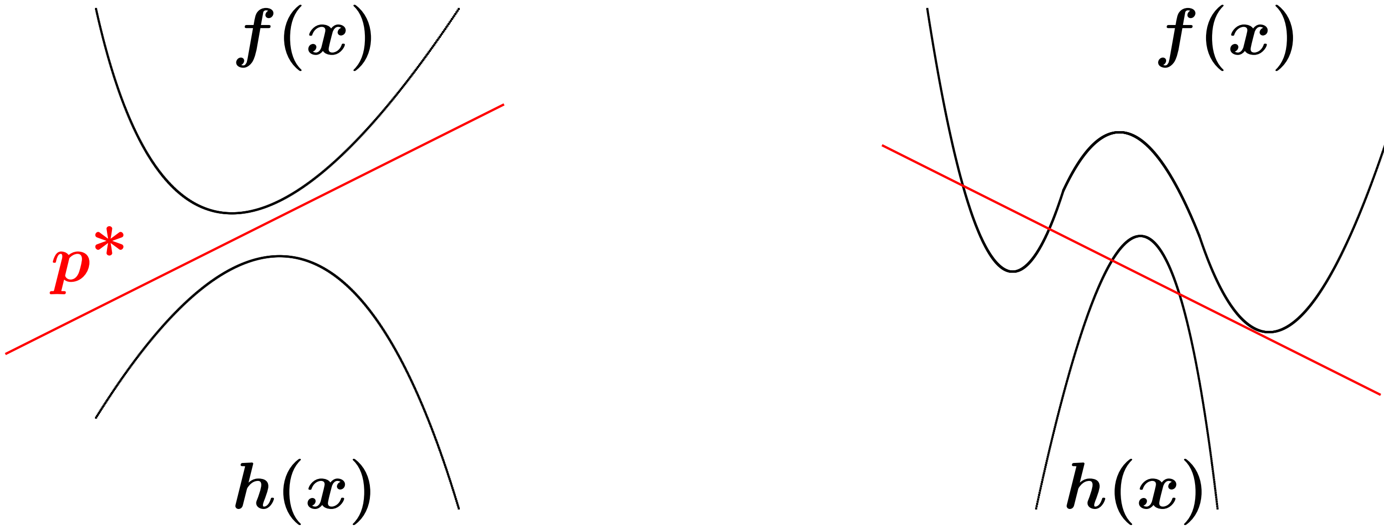


History of Discrete Conjugacy

Matroid bases	$\longleftrightarrow$	Matroid rank fn
Whitney 35		Whitney 35
$\Downarrow$		$\Downarrow$
Polymatroid	$\longleftrightarrow$	**Submodular fn**
Edmonds 70		Edmonds 70
$\Downarrow$		$\Downarrow$
Valuated matroid	|	Lovász extension
Dress–Wenzel 90	|	Lovász 83
$\Downarrow$	|	$\Downarrow$
	|	Submod. integ. conv. fn
	|	Favati-Tardella 90
		$\Downarrow$
M-convex fn	$\longleftrightarrow$	**L-convex fn**
Murota 96		Murota 98
$\Updownarrow$		$\Updownarrow$
$M^{\natural}$-convex fn	$\longleftrightarrow$	$L^{\natural}$-convex fn
Murota-Shioura 99		Fujishige–Murota 00

Duality: Separation Theorem

main issue in convexity paradigm



$f : \mathrm{R}^n \to \mathrm{R}$ **convex**

$h : \mathrm{R}^n \to \mathrm{R}$ **concave**

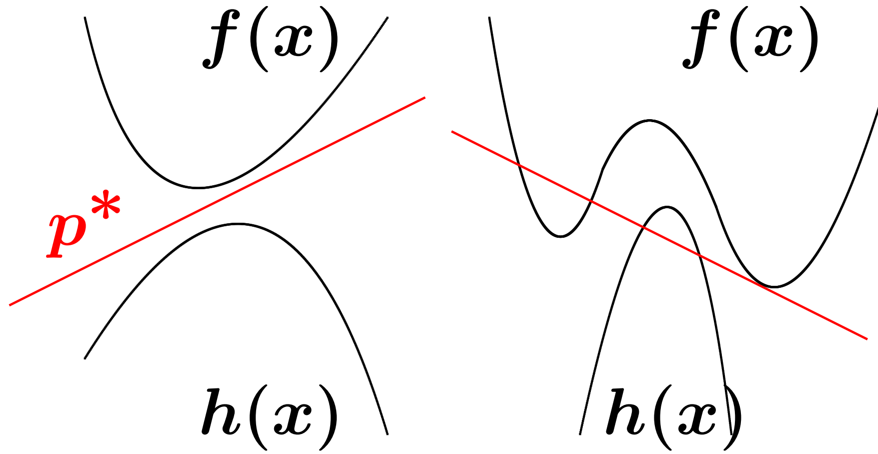
$$f(x) \geq h(x) \quad (\forall x) \;\Rightarrow\; \exists\ \alpha^* \in \mathrm{R}, \quad \exists\ p^* \in \mathrm{R}^n:$$

$$f(x) \geq \alpha^* + \langle p^*, x \rangle \geq h(x) \qquad (x \in \mathrm{R}^n)$$

Discrete Separation Theorem

$f : \mathbf{Z}^n \to \mathbf{R}$ "convex"

$h : \mathbf{Z}^n \to \mathbf{R}$ "concave"



- $f(x) \geq h(x) \quad (\forall x \in \mathbf{Z}^n) \Rightarrow \exists\ \alpha^* \in \mathbf{R}, \quad \exists\ p^* \in \mathbf{R}^n$:

$$f(x) \geq \alpha^* + \langle p^*, x \rangle \geq h(x) \qquad (x \in \mathbf{Z}^n)$$

- f, h: **integer-valued** $\Rightarrow \alpha^* \in \mathbf{Z}, \quad p^* \in \mathbf{Z}^n$

Frank's Discrete Separation

Equivalent to Edmonds' polymatroid intersection

$\rho : 2^V \to \mathrm{R}$: **submodular** $(\rho(\emptyset) = 0)$

$\mu : 2^V \to \mathrm{R}$: **supermodular** $(\mu(\emptyset) = 0)$

- $\rho(X) \geq \mu(X) \quad (\forall X \subseteq V) \Rightarrow \exists\ x^* \in \mathrm{R}^V$:

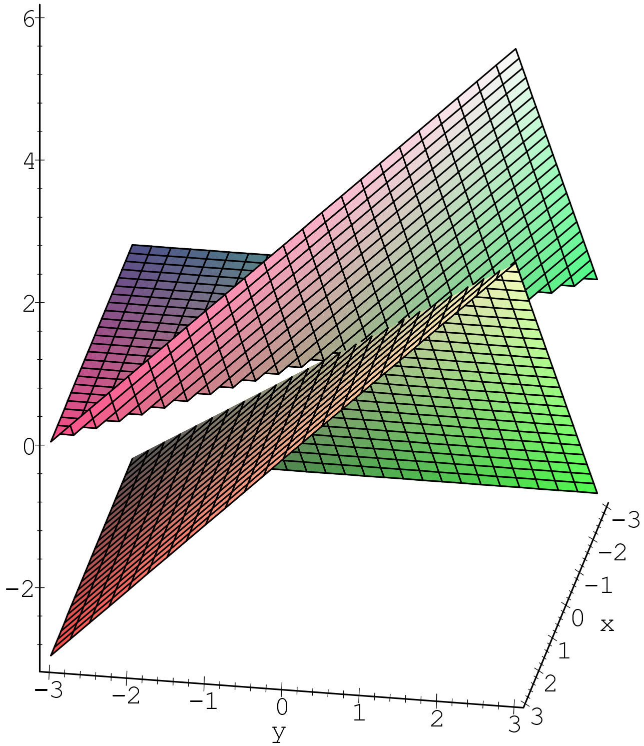
$$\rho(X) \geq x^*(X) \geq \mu(X) \qquad (\forall X \subseteq V)$$

- ρ, μ: **integer-valued** $\Rightarrow x^* \in \mathrm{Z}^V$

Difficulty of Discrete Separation (1)

$f(x, y) = \max(0, x + y)$ **convex**

$h(x, y) = \min(x, y)$ **concave**



separable but nonintegral

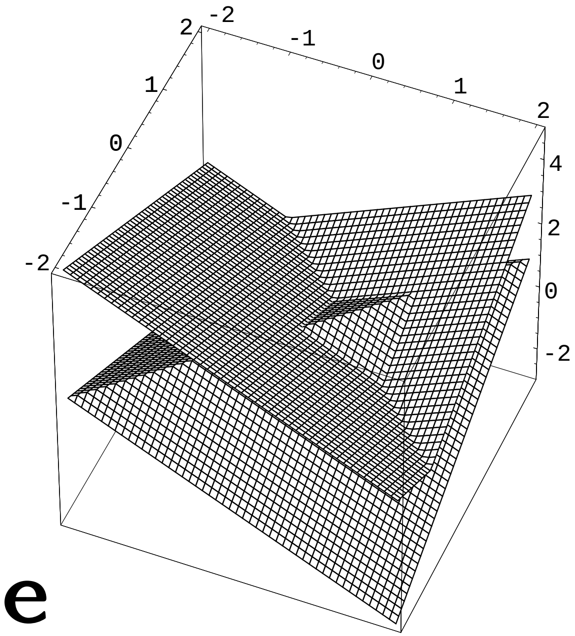
$p^* = (1/2, 1/2)$, $\alpha^* = 0$ **unique separating plane**

Difficulty of Discrete Separation (2)

Even real-separation is nontrivial

$f(x, y) = |x + y - 1|$ **convex**

$h(x, y) = 1 - |x - y|$ **concave**



- $f(x, y) \geq h(x, y)$ $(\forall (x, y) \in \mathbf{Z}^2)$ **true**
- $f(x, y) \geq h(x, y)$ $(\forall (x, y) \in \mathbf{R}^2)$ **not true**

since $f = 0 < h = 1$ **at** $(x, y) = (1/2, 1/2)$

$\Longrightarrow$ **No** $\alpha^* \in \mathbf{R}$, $p^* \in \mathbf{R}^2$ **satisfies**

$$f(x) \geq \alpha^* + \langle p^*, x \rangle \geq h(x)$$

Discrete Separation Theorems

(Murota 96/98)

M-separation Thm: M$^\natural$-convex fn

$\supset$ Weight splitting for weighted matroid intersection

(Iri-Tomizawa 76, Frank 81)

(linear fn, indicator fn $=$ M$^\natural$-convex fn)

L-separation Thm: L$^\natural$-convex fn

$\supset$ Discrete separation for submod. set function

(Frank 82)

(submod. set fn $=$ L$^\natural$-convex fn on 0–1 vectors)

Min-Max Duality

Legendre–Fenchel transform

$$f^{\bullet}(p) = \sup\{\langle p, x\rangle - f(x) \mid x \in \mathbf{Z}^n\}$$

$$h^{\circ}(p) = \inf\{\langle p, x\rangle - h(x) \mid x \in \mathbf{Z}^n\}$$

Fenchel-type duality thm

f: M$^{\natural}$-convex $\quad$ h: M$^{\natural}$-concave $\quad$ ($\mathbf{Z}^n \to \mathbf{Z}$)

$$\inf_{x \in \mathbf{Z}^n} \{f(x) - h(x)\} = \sup_{p \in \mathbf{Z}^n} \{h^{\circ}(p) - f^{\bullet}(p)\}$$

self-conjugate $\quad$ ($f^{\bullet}$: L$^{\natural}$-conv $\quad$ h°: L$^{\natural}$-conv)

$\Longrightarrow$ Edmonds' matroid intersection thm

Relation among Duality Thms

Discrete Convex	Combinatorial Opt.
M-separation	
$f(x) \geq \boxed{\textsf{Lin}} \geq h(x)$	Fenchel duality (Fujishige 84) matroid intersect. (Edmonds 70)
$\Updownarrow$	$\Updownarrow$
Fenchel duality $\inf\{f - h\}$ $= \sup\{h^\circ - f^\bullet\}$	$\Rightarrow$ discrete separ. for submod (Frank 82) $\Rightarrow$ valuated matroid intersect. (M. 96)
$\Updownarrow$	$\Downarrow$
L-separation $f^\bullet(p) \geq \boxed{\textsf{Lin}} \geq h^\circ(p)$	weighted matroid intersect. (Edmonds 79, Iri-Tomizawa 76, Frank 81)

A.
Minimization (General)

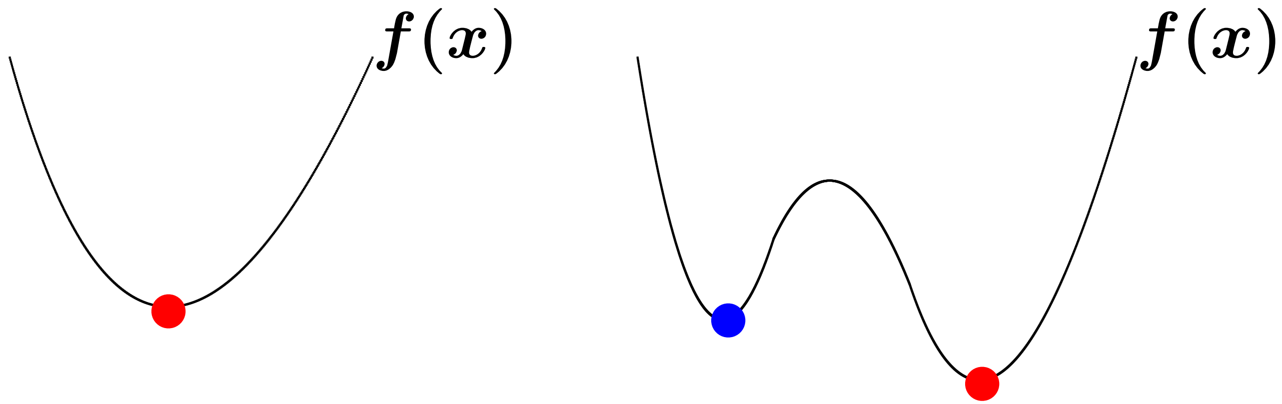
Optimality Criterion

Descent Method

Scaling and Proximity

Optimality Criterion

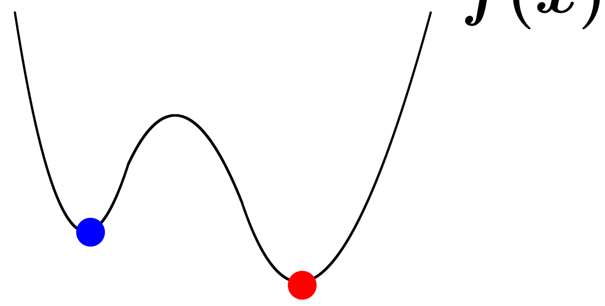
Global opt vs Local opt



convex nonconvex

Local opt wrt neighborhood

Descent Method



S0: Initial sol x^*

S1: Minimize $f(x)$ in nbhd of x^* to obtain $x^\bullet$

S2: If $f(x^*) \leq f(x^\bullet)$, return x^* (local opt)

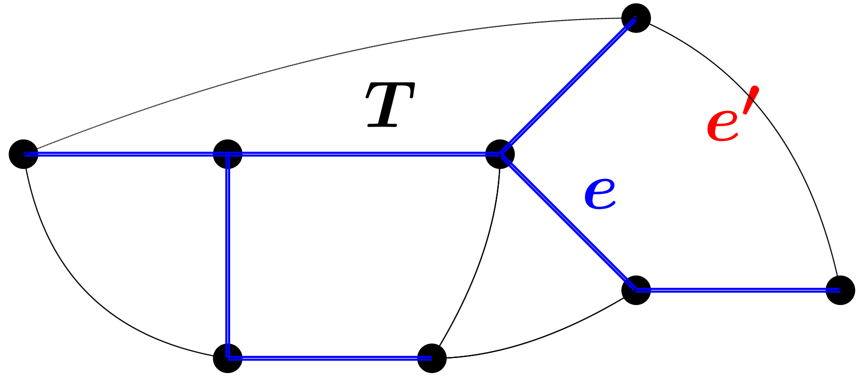
S3: Update $x^* = x^\bullet$; go to S1

······What is nbhd ?

A1.

M-convex Minimization

Min Spanning Tree Problem



length $d : E \to \mathrm{R}$

total length of T

$$\tilde{d}(T) = \sum_{e \in T} d(e)$$

Thm

T: **MST** $\iff \tilde{d}(T) \leq \tilde{d}(T - e + e')$

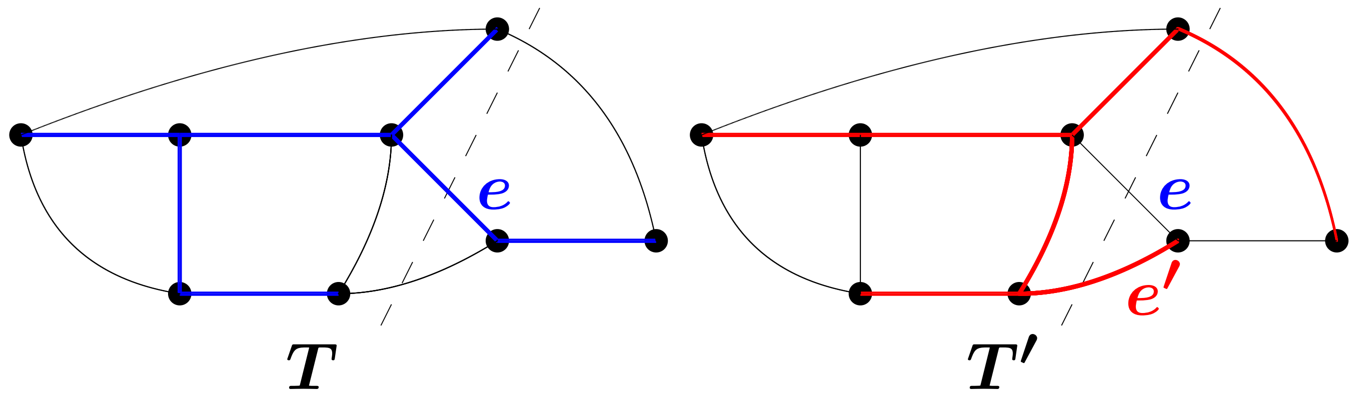
$\iff d(e) \leq d(e')$ **if** $T - e + e'$ **is tree**

Algorithm **Kruskal's, Kaleba's**

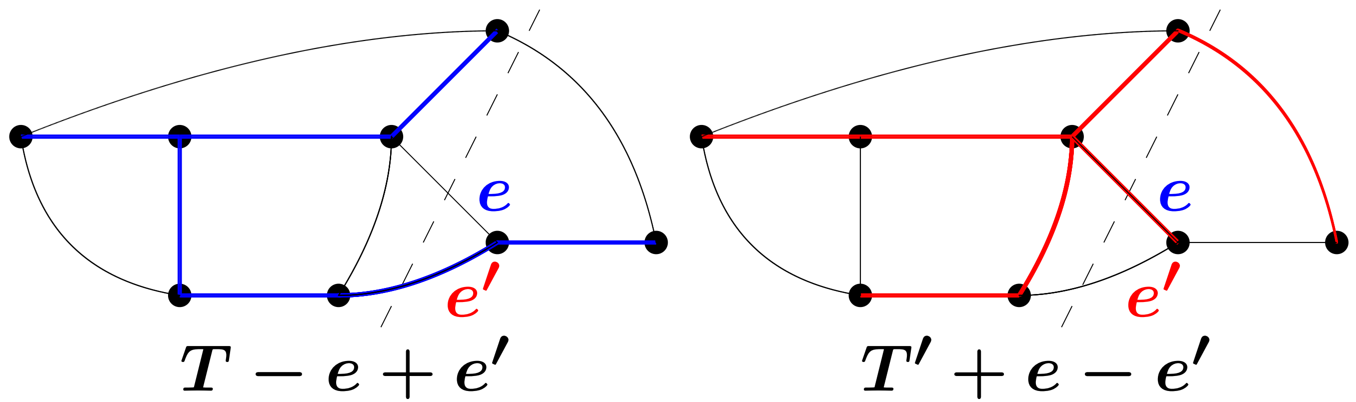
DCA view

- **linear optimization on an M-convex set**
- **M-optimality:** $f(x^*) \leq f(x^* - e_i + e_j)$

Tree: Exchange Property



Given pair of trees



New pair of trees

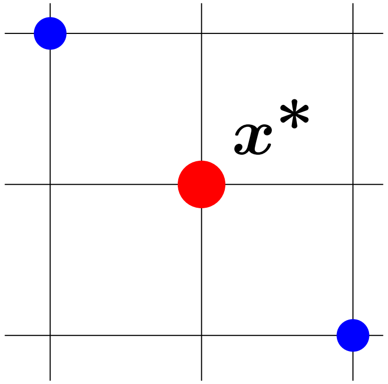
Exchange property: For any $T, T' \in \mathcal{T}$, $e \in T \setminus T'$ there exists $e' \in T' \setminus T$ s.t. $T - e + e' \in \mathcal{T}$, $T' + e - e' \in \mathcal{T}$

Local vs Global Opt (M-conv)

Thm : $f : \mathbb{Z}^n \to \mathbb{R}$ M-convex (Murota 96)

x^*: global opt

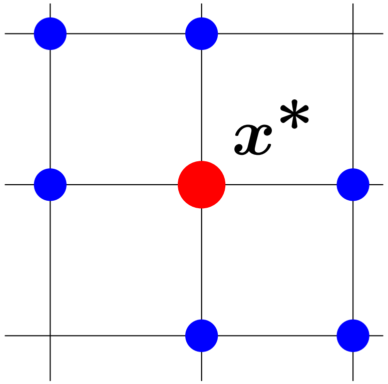
$\iff$ local opt $f(x^*) \le f(x^* - e_i + e_j)$ $(\forall i, j)$



Ex: $x^* + (0, 1, 0, 0, -1, 0, 0, 0)$

Can check with n^2 fn evals

For M♮-convex fn $\Rightarrow$



Steepest Descent for M-convex Fn

S0: Find a vector $x \in \mathrm{dom} f$

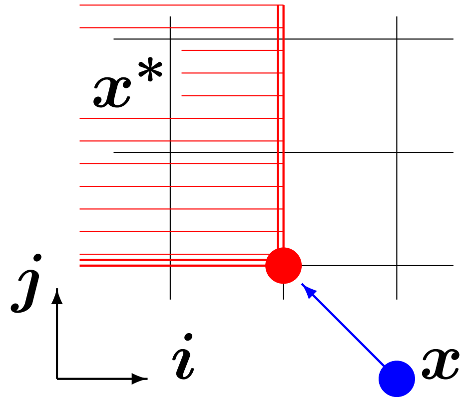
S1: Find $i \neq j$ that minimize $f(x - e_i + e_j)$

S2: If $f(x) \leq f(x - e_i + e_j)$, stop

(x: minimizer)

S3: Set $x := x - e_i + e_j$

and go to S1



Minimizer Cut Thm (Shioura 98)

$\exists$ minimizer x^* with $x^*_i \leq x_i - 1, \quad x^*_j \geq x_j + 1$

$\Rightarrow$ Murota 03, Shioura 98, 03, Tamura 05

- Kalaba's for min spannning tree
- Dress–Wenzel's for valuated matroid

A2.

L-convex Minimization

Shortest Path Problem (one-to-all)

one vertex (s) to all vertices, length $\ell \geq 0$, integer

Dual LP

Maximize	$\Sigma\ p(v)$
subject to	$p(v) - p(u) \leq \ell(u,v)\ \forall(u,v)$
	$p(s) = 0$

Algorithm Dijkstra's

DCA view

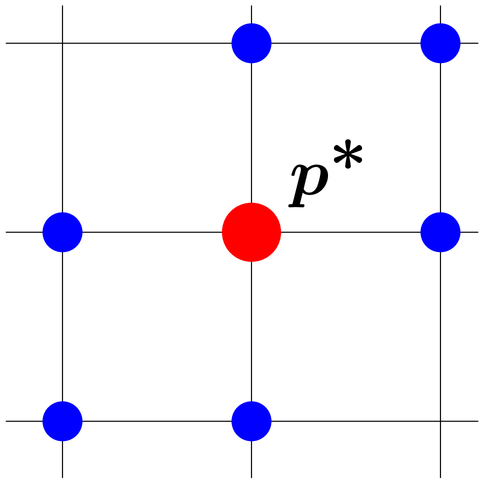
- linear optimization on an L$^\natural$-convex set (in polyhedral description)
- Dijkstra's algorithm (M.-Shioura 12)
  $=$ steepest ascent for L$^\natural$-concave maximization
  with uniform linear objective $(1, 1, \ldots, 1)$

Local vs Global Opt (L♮-conv)

Thm : $g : \mathrm{Z}^n \to \mathrm{R}$ L♮-convex (Murota 98,03)

p^*: global opt

$\Longleftrightarrow$ local opt $g(p^*) \le g(p^* \pm q)$ $(\forall q \in \{0,1\}^n)$



Ex: $p^* + (0,1,0,1,1,1,0,0)$

$\Longleftrightarrow$ $\rho_{\pm}(X) = g(p^* \pm \chi_X) - g(p^*)$
takes min at $X = \emptyset$

Can check with n^5 (or less) fn evals
using submodular fn min algorithm
(Iwata-Fleischer-Fujishige, Schrijver, Orlin)

Steepest Descent for L$^\natural$-convex Fn

(Iwata 99, Murota 00, 03, Kolmogorov-Shioura 09)

S0: Find a vector $p \in \mathrm{dom} g$

S1: Find $\varepsilon = \pm 1$ and X that $\boxed{\text{minimize } g(p + \varepsilon \chi_X)}$

S2: If $g(p) \leq g(p + \varepsilon \chi_X)$, stop ($p$: minimizer)

S3: Set $p := p + \varepsilon \chi_X$ and go to S1

- Dijkstra's algorithm for shortest path (M.-Shioura 12)

  π: potential, $V \setminus U$: permanent labeled

  Special case with $g(p) = -1^\top p$:

  $\pi(v) = \min\{p(u) + \ell(u, v) \mid u \notin U\} \quad (v \in U \setminus \{s\})$

C1.
Electric Circuit

Combinatorics on top of convexity

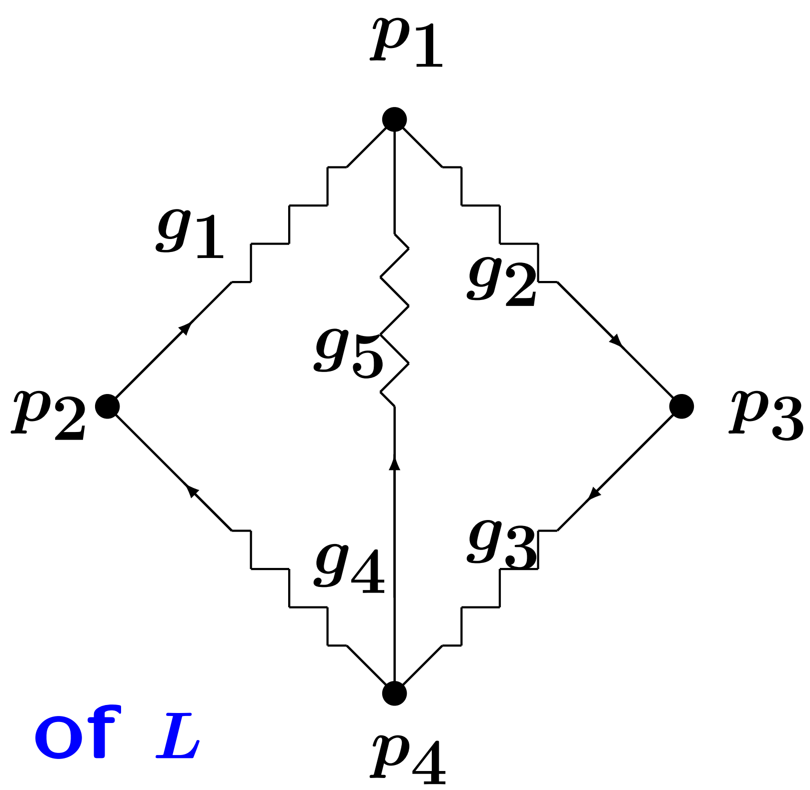
Linear Resistor Circuit

g_j: conductance

p: potential at nodes

Consumed power:

$$g(p) = \frac{1}{2}\, p^\top L p$$

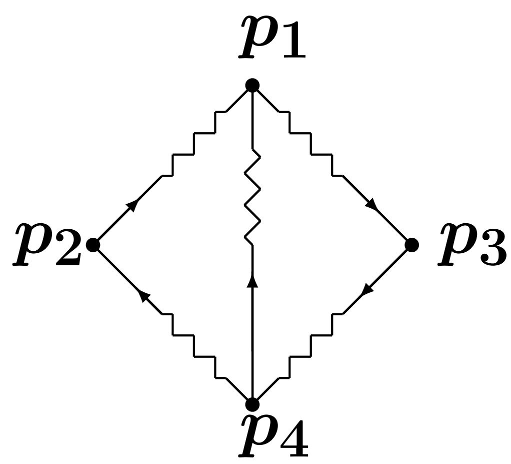


Combinatorial property of L

- off-diagonal $\quad \ell_{ij} \leq 0 \qquad (i \neq j; 1 \leq i, j \leq n)$
- row-sum $\quad \sum_{j=1}^{n} \ell_{ij} \geq 0 \qquad (1 \leq i \leq n)$

Node-branch incidende **(graph structure)**

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 \end{bmatrix}$$



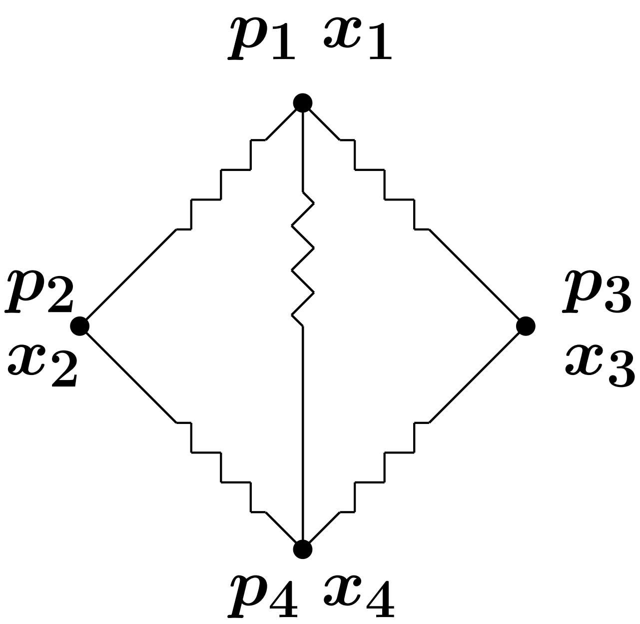
Node admittance

$$L = A \ \mathrm{diag}\,(g_j) \ A^\top =$$

$$\begin{bmatrix} g_1 + g_2 + g_5 & -g_1 & -g_2 & -g_5 \\ -g_1 & g_1 + g_4 & 0 & -g_4 \\ -g_2 & 0 & g_2 + g_3 & -g_3 \\ -g_5 & -g_4 & -g_3 & g_3 + g_4 + g_5 \end{bmatrix}$$

(off-diag ≤ 0, row sum $= 0$)

Nonlinear Resistor Circuit



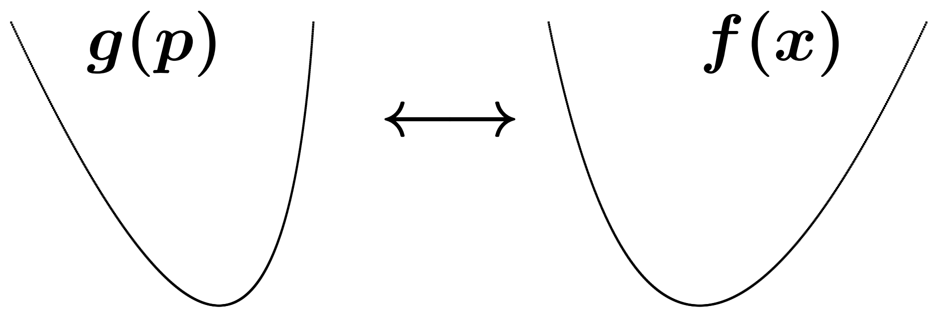
discrete structure	+	convex fn
electric circuit	+	power

voltage source p

$g(p)$: voltage potential (power)

current source x

$f(x)$: current potential (power)



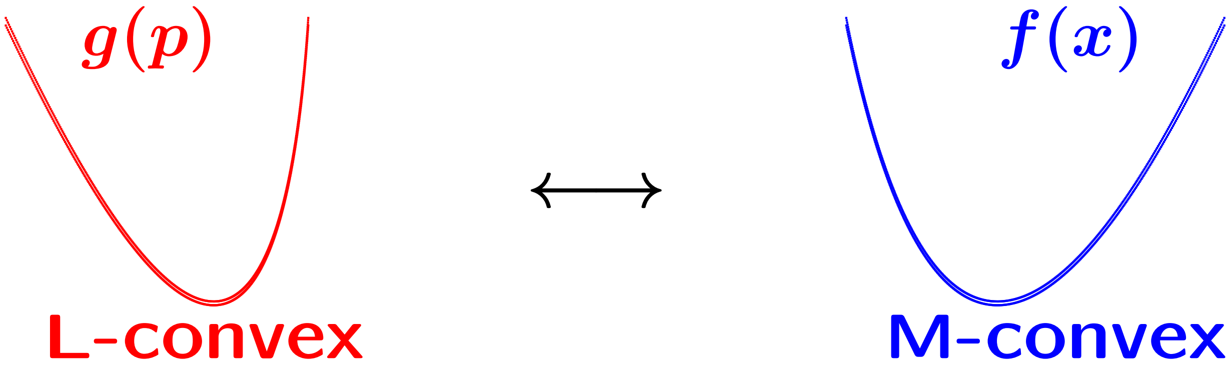
- $g(p)$ and $f(x)$ are convex
- $g(p)$ and $f(x)$ are conjugate (Legendre transform)

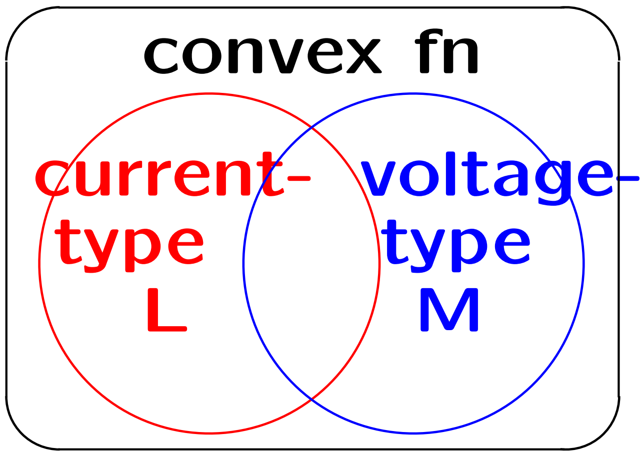
$\Longrightarrow$ **What is their difference?**

Distinction btw Voltage and Current

discrete structure + convexity

voltage p, power $g(p)$ **current x, power $f(x)$**





C2.

L-/M-convexity in Continuous Variables

L$^\natural$-convex Fn in Continuous Vars

(Murota–Shioura 00, 03)

Continuous $\mathrm{R}^n \to \mathrm{R}$		Discrete $\mathrm{Z}^n \to \mathrm{R}$
convexity		
$\Updownarrow$ (convex fn)	discr	
mid-pt convexity	$\longrightarrow$	discr mid-pt convexity
		$\Updownarrow$ (L$^\natural$-convex fn)
transl. submodular	$\longleftarrow$	transl. submodular
(L$^\natural$-convex fn)	conti	

mid-pt conv: $f(x) + f(y) \geq f\left(\frac{x+y}{2}\right) + f\left(\frac{x+y}{2}\right)$

discr mid-pt conv: $f(x) + f(y) \geq f\left(\left\lceil\frac{x+y}{2}\right\rceil\right) + f\left(\left\lfloor\frac{x+y}{2}\right\rfloor\right)$

transl. sbm: $f(x)+f(y) \geq f((x-\alpha 1)\vee y)+f(x\wedge(y+\alpha 1))$

continuous $\alpha \in \mathrm{R}_+$ $\longleftarrow$ discrete $\alpha \in \mathrm{Z}_+$

M$^\natural$-convex Fn in Continuous Vars

(Murota–Shioura 00, 03)

Continuous $\mathbf{R}^n \to \mathbf{R}$		Discrete $\mathbf{Z}^n \to \mathbf{R}$
convexity		
$\Updownarrow$ (convex fn)	discr	
equi-dist convexity	$\longrightarrow$	
		exchange property
exchange property	$\longleftarrow$	
(M$^\natural$-convex fn)	conti	(M$^\natural$-convex fn)

equi-distance convex:

$$f(x) + f(y) \geq f(x - \alpha(x-y)) + f(y + \alpha(x-y))$$

exchange: $f(x)+f(y) \geq \min \Big[f(x - \alpha e_i) + f(y + \alpha e_i)$

$$\min_{x_j < y_j} \{ f(x - \alpha(e_i - e_j)) + f(y + \alpha(e_i - e_j)) \} \Big]$$

continuous $\alpha \in \mathbf{R}_+$ $\longleftarrow$ discrete $\alpha \in \mathbf{Z}_+$

M-L Conjugacy Theorem

$f : \mathbf{Z}^n \to \overline{\mathbf{Z}}$, $\quad f : \mathbf{R}^n \to \overline{\mathbf{R}}$

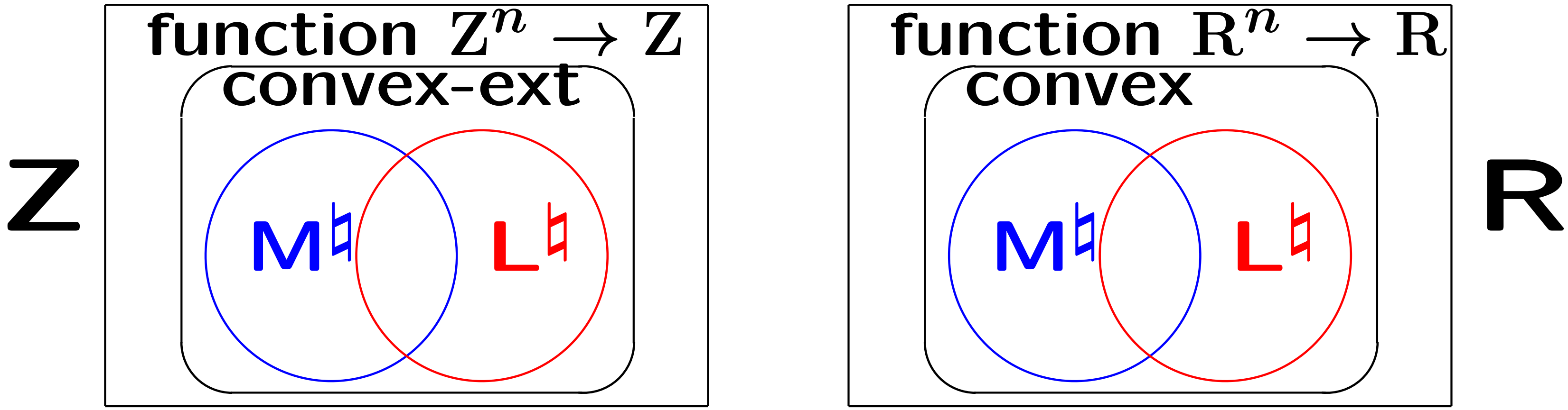
Legendre transform: $f^\bullet(p) = \sup_x [\langle p, x \rangle - f(x)]$

M$^\natural$-convex and L$^\natural$-convex are conjugate

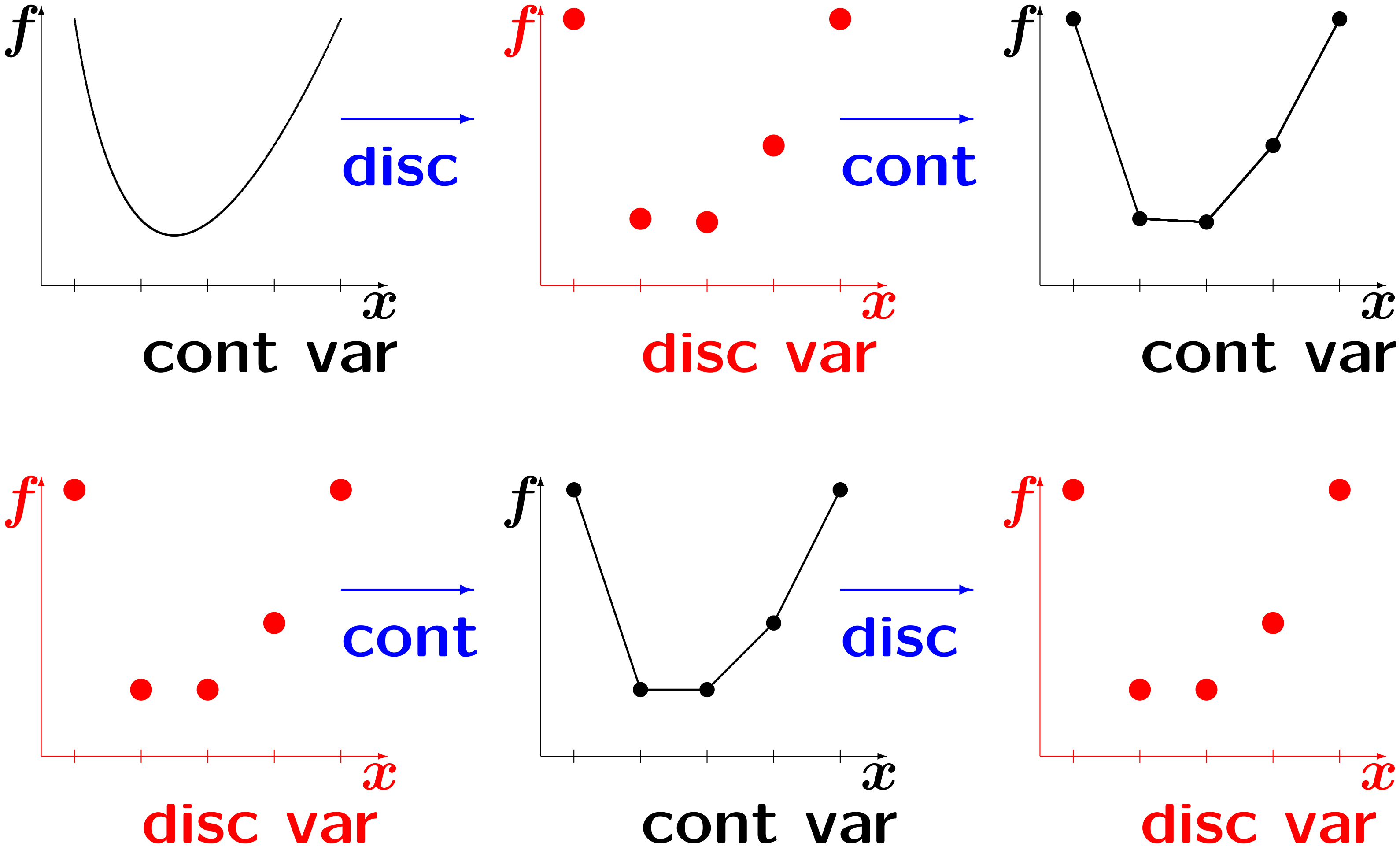
(1) $\mathbf{Z}^n \to \mathbf{Z}$ (Murota 98)

(2) $\mathbf{R}^n \to \mathbf{R}$, polyhedral (Murota–Shioura 00)

(3) $\mathbf{R}^n \to \mathbf{R}$, closed proper (Murota–Shioura 03)



Discretization vs Prolongation



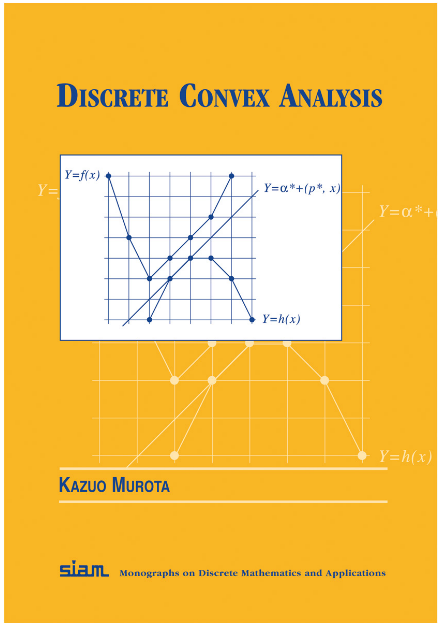
Prolongation, Discretization, Scaling

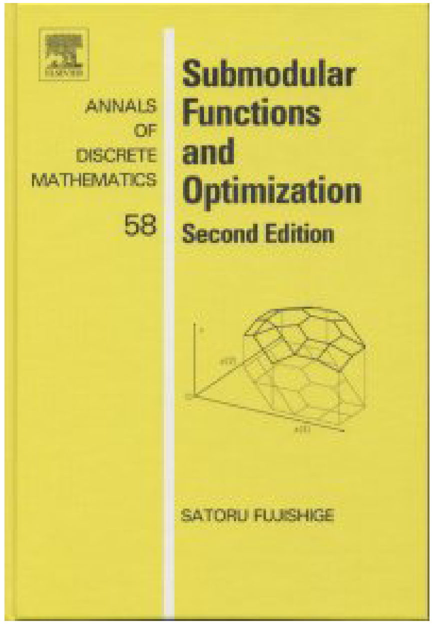
	Prolong. $\mathbf{Z}^n \Rightarrow \mathbf{R}^n$	Discret. $\mathbf{R}^n \Rightarrow \mathbf{Z}^n$	Scaling $\mathbf{Z}^n \Rightarrow (2\mathbf{Z})^n$
$\mathrm{L}^\natural$-conv	OK	OK	OK
$\mathrm{M}^\natural$-conv	OK	NG	NG
(quadratic)	OK′	NG	OK
(laminar)	OK	OK	OK

Books/Surveys

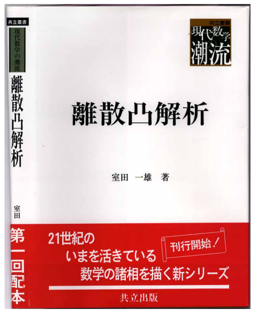
Murota: Discrete Convex Analysis, SIAM, 2003

Fujishige: Submodular Functions and Optimization, 2nd ed., Elsevier, 2005 (Chap. VII)

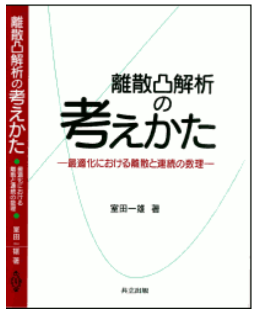




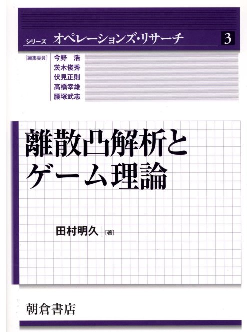
M. 01



M. 07



Tamura 09



Murota: Recent developments in discrete convex analysis, in: Research Trends in Combinatorial Optimization, Bonn 2008, Springer, 2009, Chap. 11, 219–260.

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