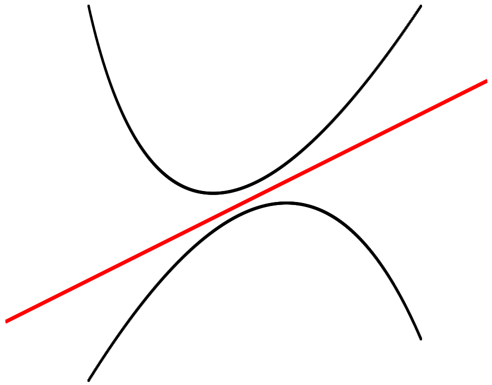
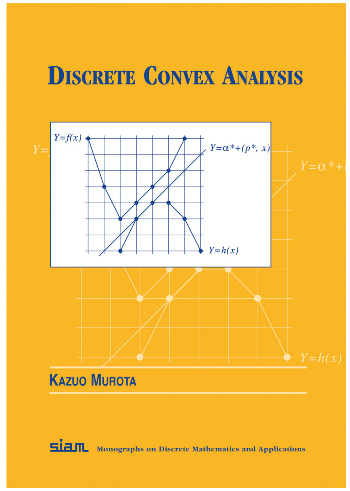
M-convex Functions on Jump Systems — A Survey —

Kazuo Murota (U. Tokyo)





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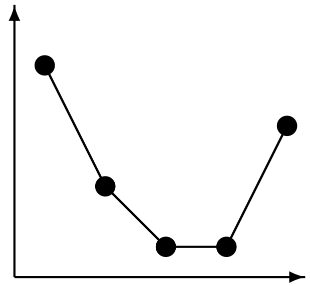
Discrete Convex Analysis

Convexity Paradigm in Discrete Optimization

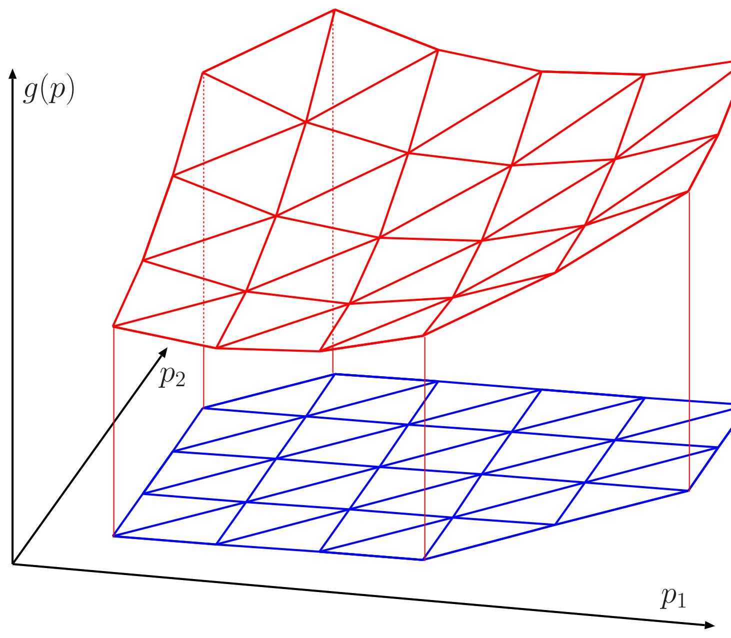
Matroid Theory + Convex Analysis

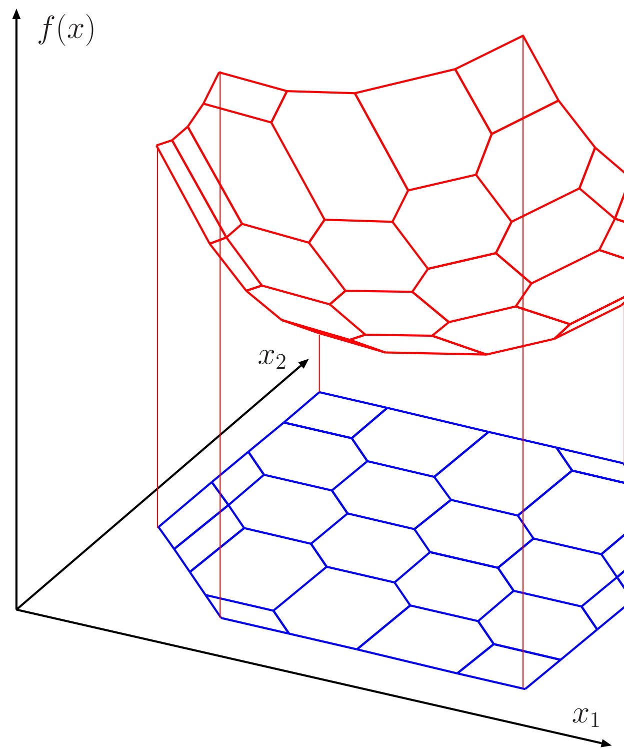
Submodular fn	$\leftrightarrow$	Matroid base
$\Downarrow$		$\Downarrow$
L-convex fn	$\leftrightarrow$	M-convex fn



- Global optimality $\Longleftrightarrow$ local optimality
- Conjugacy: Legendre–Fenchel transform
- Duality (Fenchel min-max, discrete separation)
- Minimization algorithms
- Applications: OR, game, economics, matrices

Discrete Convex Functions





L$^\natural$-convex fn **M$^\natural$-convex fn**

General Understanding

Poly. Solvable $\approx$ Discrete Convex

Linear/Convex Functions on $\bullet\bullet\bullet\bullet\bullet$

- Matroids / Polymatroids / Base Polyhedra

  spanning tree, bipartite matching,

  min-cost flow

- Jump Systems

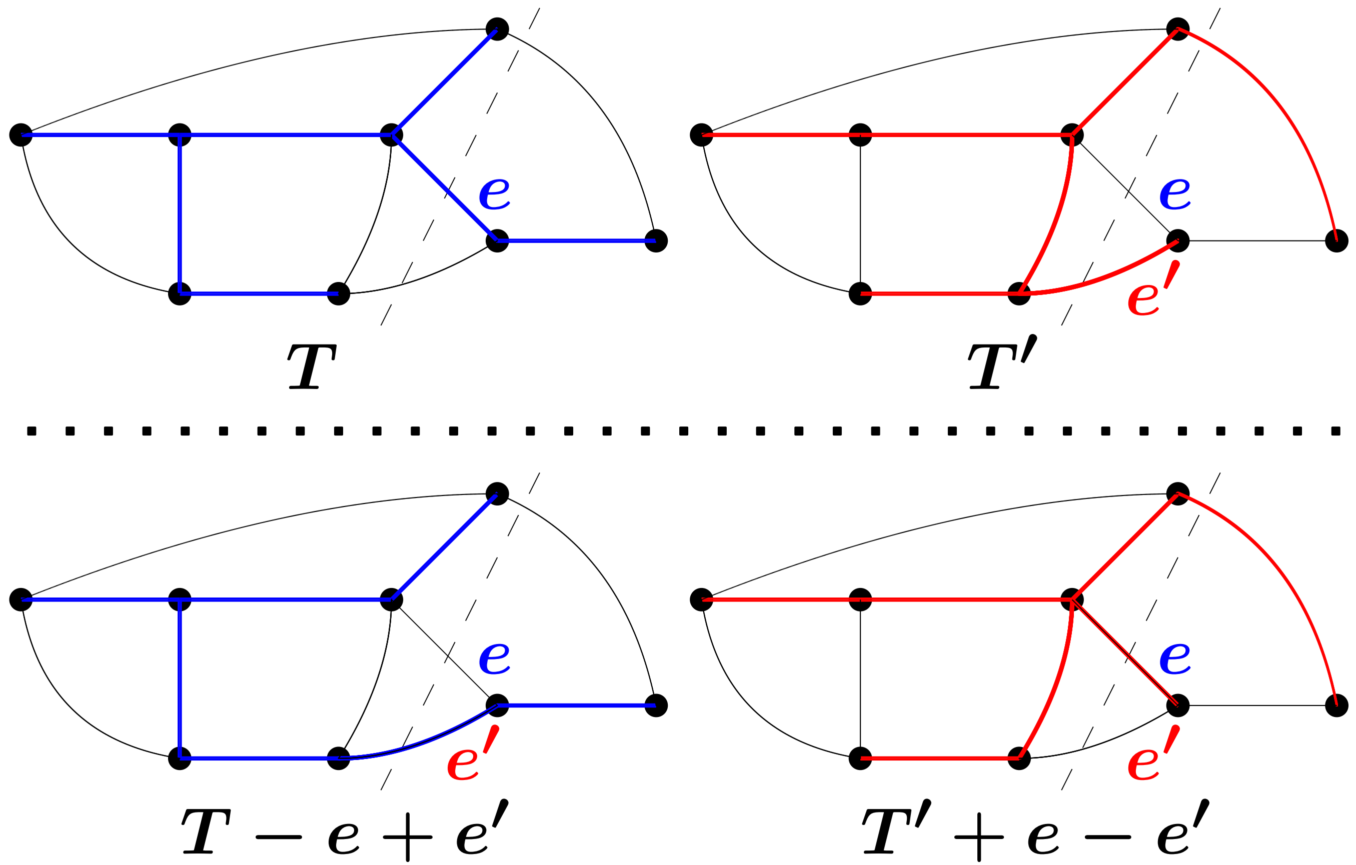
  nonbipartite matching, factor

Contents

J1.

Jump System

Tree: Exchange Property



Given pair of trees

New pair of trees

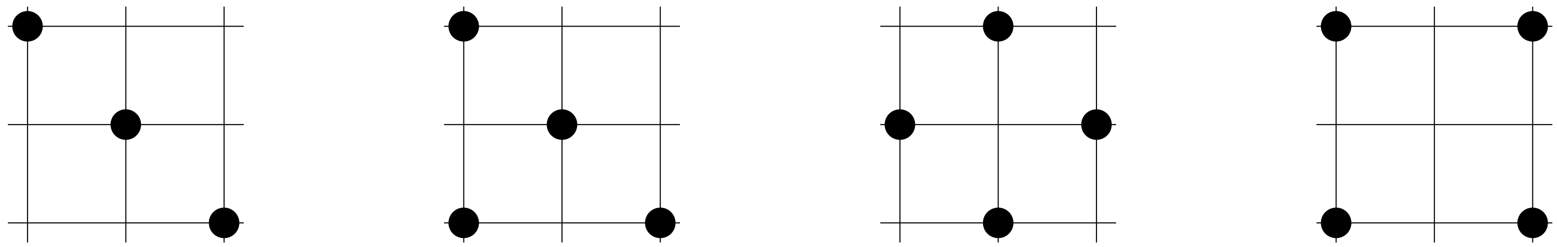
Exchange property: For any $T, T' \in \mathcal{T}$, $e \in T \setminus T'$ there exists $e' \in T' \setminus T$ s.t. $T - e + e' \in \mathcal{T}$, $T' + e - e' \in \mathcal{T}$

Jump Systems

Bouchet–Cunningham (95) SIAM	"jump system"
Sebő (95)	observations on gap, sum, etc.
Geelen (96) WEB	observations on exchange, etc.
Lovász (97) JCTB	membership problem
Cunningham (02) MP	solvability=jump
Kabadi–Sridhar (05) JAMDC	relation to Δ-matroid
Szabó (08) SIAM	conjecture of Recski

Jump System

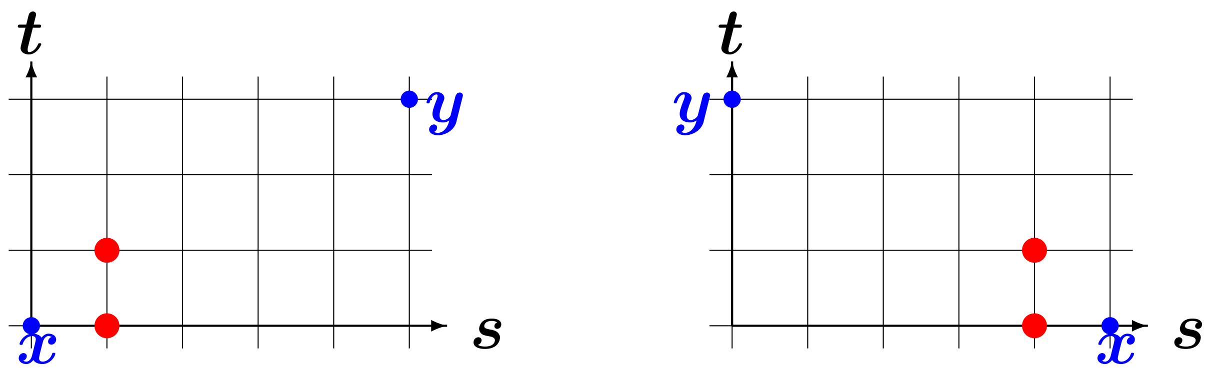
Bouchet-Cunningham (95)



J: jump system $\iff$ (2-step axiom)

$\forall x, y \in J$, $\forall$ (x, y)-incr s, either $x + s \in J$

or else $\exists$ $(x + s, y)$-incr t s.t. $x + s + t \in J$

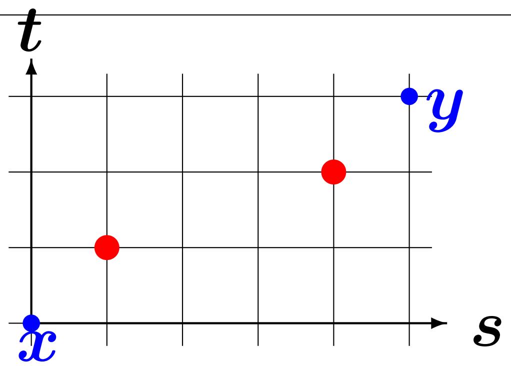


Constant-Parity Jump System

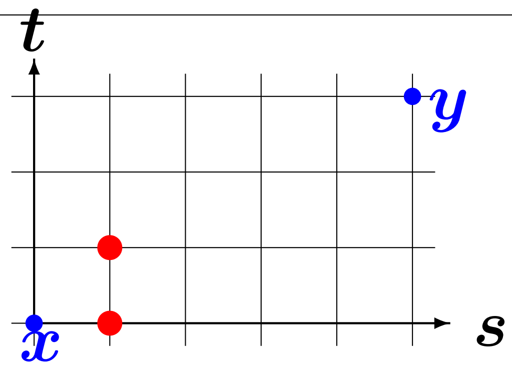
J: **const-parity jump system** **(Geelen)**

$\iff$ $\forall x, y \in J$, $\forall$ (x, y)-incr s, $\exists$ $(x+s, y)$-incr t

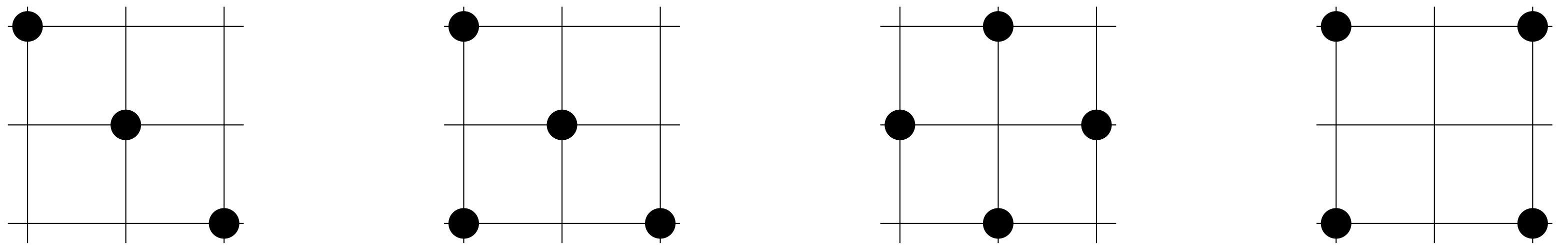
s.t. $x+s+t,\ y-s-t \in J$



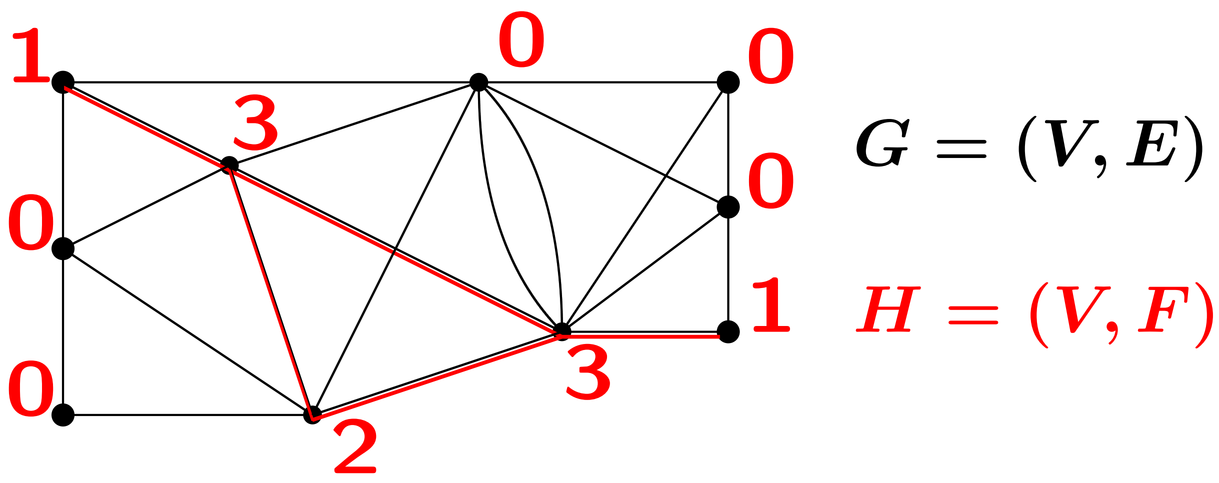
simultaneous exchange



2-step axiom



Degree System



Degree sequence: $\deg_H \in \mathbb{Z}^V$

$\deg_H(v) =$ **# edges incident to** v

$J = \{\deg_H \mid H \subseteq G\}$: **const-parity jump system**

Subclasses of Jump Systems

const-sum jump	$\longrightarrow$	(projection)
$\vert\vert\vert$	[Fujishige]	
base polyhedron	$\simeq$	generalized polymatroid
	[Geelen]	
const-parity jump	$\simeq$	simul. exchange jump
	[Wenzel, Duchamp]	
even Δ-matroid	$\simeq$	simul. exch. Δ-matroid

J: simul. exchange jump system $\iff$ (SE axiom)

$\forall x, y \in J$, $\forall$ (x, y)-incr s, either $x + s,\ y - s \in J$

or else $\exists$ $(x + s, y)$-incr t s.t. $x + s + t,\ y - s - t \in J$

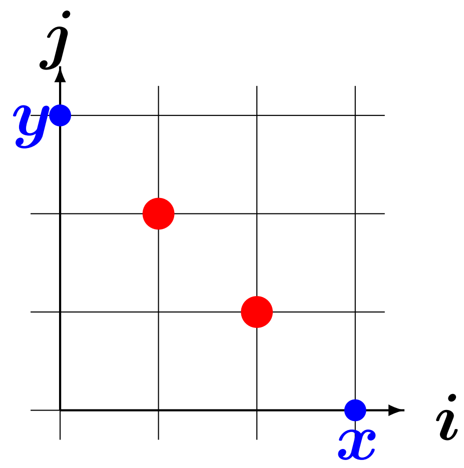
J2.
M-convex Function

(M = Matroid)

M-convex Function on Base Polyhedra

$f : \mathbf{Z}^n \to \mathbf{R} \cup \{+\infty\}$

e_i: i-th unit vector

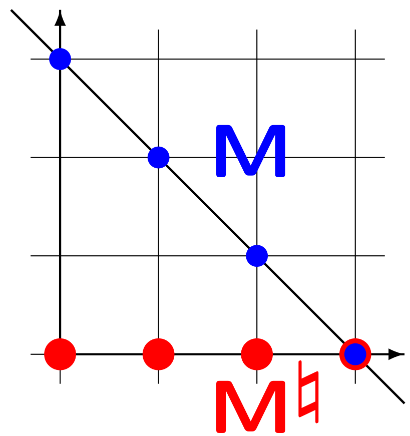


Def: f is M-convex

$\Longleftrightarrow \quad \forall x, y, \quad \forall i : x_i > y_i, \quad \exists j : x_j < y_j$:

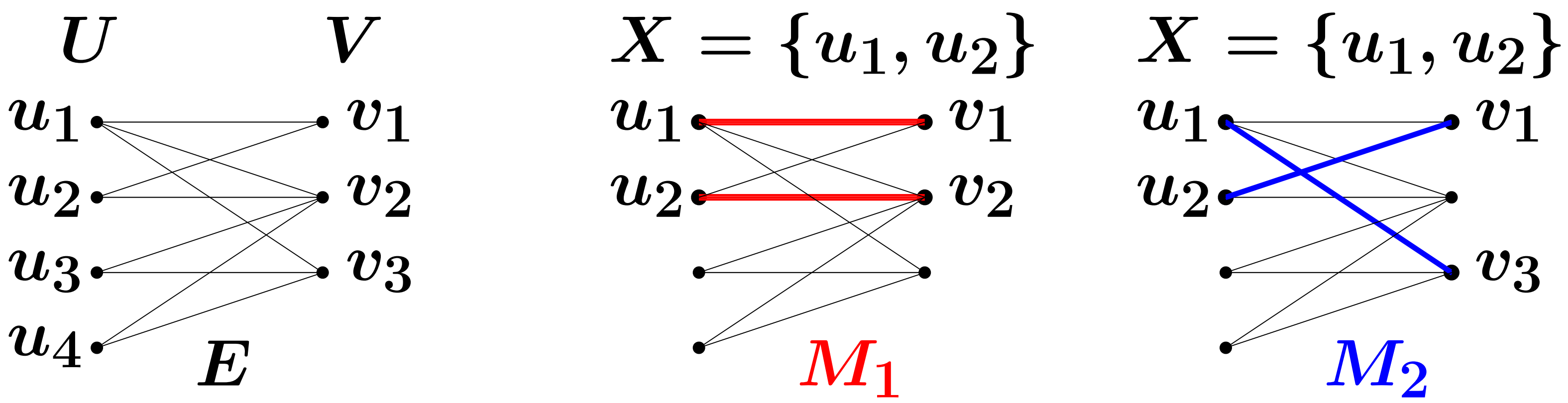
$$f(x) + f(y) \geq f(x - e_i + e_j) + f(y + e_i - e_j)$$

$\mathrm{dom} f \subseteq$ const-sum hyperplane



$$\mathrm{M}_{n+1} \simeq \mathrm{M}^{\natural}_{n} \supsetneqq \mathrm{M}_{n}$$

Matching / Assignment



Max weight for $X \subseteq U$ (w: given weight)

$$f(X) = \max\{\sum_{e \in M} w(e) \mid M\text{: matching, } U \cap \partial M = X\}$$

Max-weight function f is M$^\natural$-concave (Murota 96)

- Proof by augmenting path
- Extension to min-cost network flow

Convex Functions on Jump Systems

Ando–Fujishige–Naitoh (95) JORSJ	separable convex
Murota (06) SIAM	"M-convex function"
Murota–Tanaka (06) IEICE	descent alg for M-convex
Shioura–Tanaka (07) SIAM	polynom alg for M-convex
Kobayashi–Murota–Tanaka (07) SIAM	operations on M-convex
Kobayashi–Murota (07) DAM	transform. by linking
Kobayashi–Takazawa (09) JCTB	weighted even factor
Kobayashi (10) DO	C_3-free 2-matching
Kobayashi–Szabo–Takazawa (12) JCTB	C_k-free 2-matching
Berczi–Kobayashi (12) JCTB	$(n-3)$ connectivity augment.
Brändén (10) SIAM	polynomial w/ half-plane property

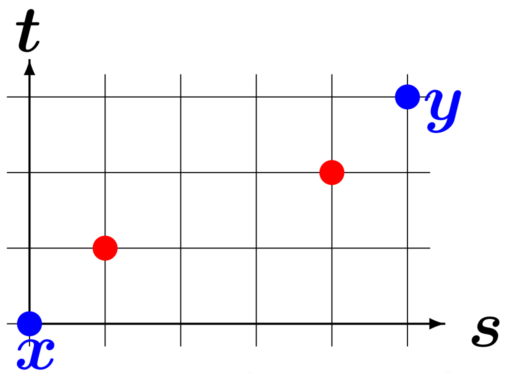
M-convex Function on Jump Systems

J: const-parity jump system, $\quad f : J \to \mathbf{R}$

(M-EXC) $\forall\ x, y \in J$ and (x,y)-incr s,

$\exists\ (x+s, y)$-incr t s.t. $x+s+t,\ y-s-t \in J$,

$$f(x) + f(y) \geq f(x+s+t) + f(y-s-t)$$

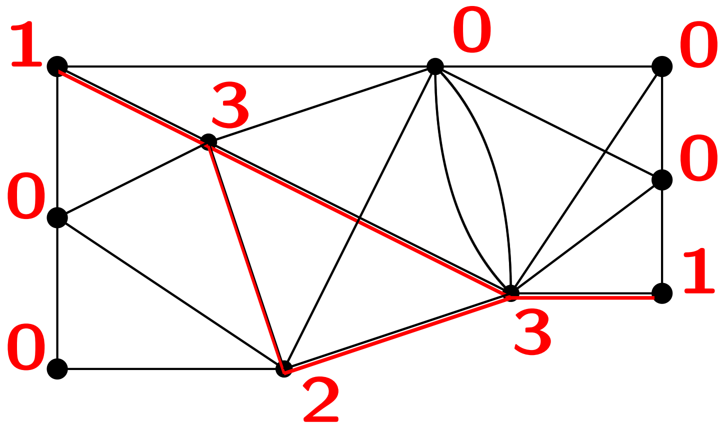


- local opt $\Longrightarrow$ global opt
- transformation by graphs
- NO integral duality

- Valuated delta matroid (Dress–Wenzel 91)
- Minsquare factor (Apollonio–Sebő 04)
- Even factor (Kobayashi–Takazawa 09)

Minsquare Factor Problem

Apollonio and Sebő 04



$G = (V, E)$

$H = (V, F)$

Degree sequence: $\deg_H \in \mathbf{Z}^V$

Problem: • given: number of edges k

Minimize $f(x) = \sum_{v \in V} (x(v))^2$

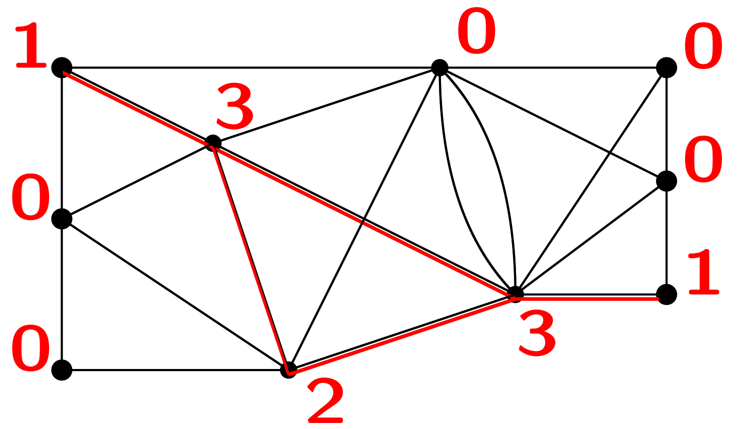
s.t. $x = \deg_H$

H has k edges $\left(\Leftrightarrow \sum_{v \in V} x(v) = 2k \right)$

M-convex minimization under const-sum constraint

Edge-Weighted Factor Problem

edge cost $w : E \to \mathrm{R}$, x: degree sequence



Problem: (cost on edges) (cost on vertices)

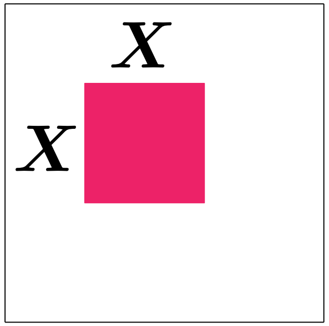
Minimize $w(F) + \sum_{v \in V} \varphi_v(x(v))$

s.t. $x = \deg_H$, H has k edges

$$f(x) = \min_H \{ w(F) \mid \deg_H = x \} + \sum_{v \in V} \varphi_v(x(v))$$

is M-convex

Skew-Symmetric Polynomial Matrix

$A =$ 

$a_{ji} = -a_{ij}$

$\{X \mid \det A[X, X] \neq 0\}$:

even delta-matroid (const-parity jump)

Bouchet 87, Chandrasekaran-Kabadi 88, Dress-Havel 86

$f(X) = \deg \det A[X, X]$:

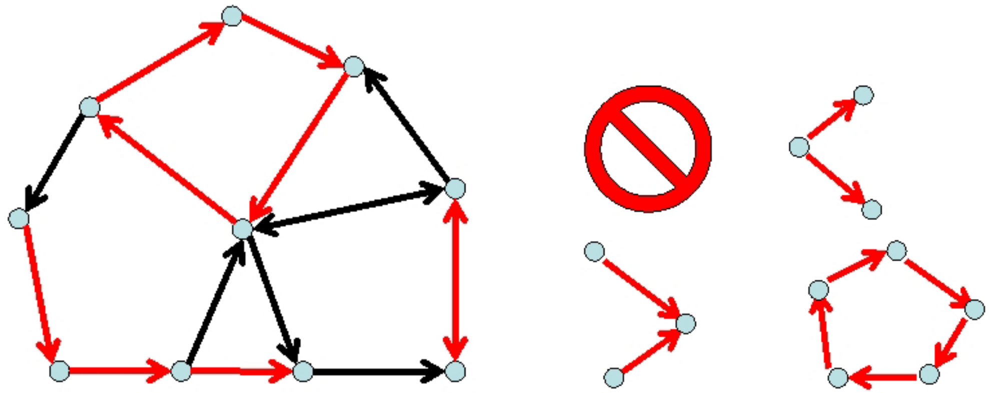
valuated delta-matroid (M-concave fn)

Dress–Wenzel 91

Even Factor

Cunningham–Geelen 01

vertex-disjoint
- paths
- even-length cycles



#: Find F with max number of arcs $|F|$

w: Find F with max arc-weight $w(F)$

- NP-hard in general
- Polynomially solvable if odd-cycle symmetric

  #:Cunningham-Geelen 01, Pap 04/07, Harvey 06

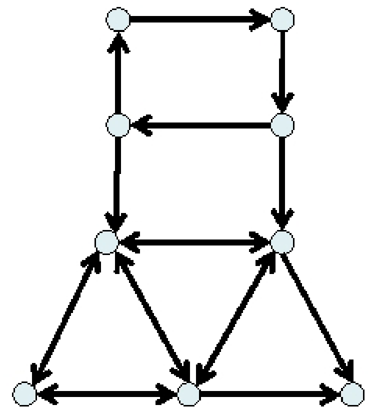
  w:Cunningham-Geelen 01, Takazawa 05/08, Király–Makai 04

Odd-Cycle Symmetry vs Convexity

Odd-cycle symmetric (OCS) $\Longleftrightarrow$

$\#$: $\forall$ odd cycle C, $\exists$ reverse cycle $\overline{C}$

w: $\forall$ odd cycle C, $w(C) = w(\overline{C})$



$d(F) = \text{outdeg}(F) \oplus \text{indeg}(F)$

$J(G) = \{d(F) \mid F : \text{even factor}\}$

$f(x) = \max\{w(F) \mid d(F) = x, \ F : \text{even factor}\}$

Thm (Kobayashi–Takazawa 09)

$\#$: G is OCS $\Longleftrightarrow$ $J(G)$ is jump system

w: (G, w) is OCS $\Longleftrightarrow$ $f(x)$ is M-concave on $J(G)$

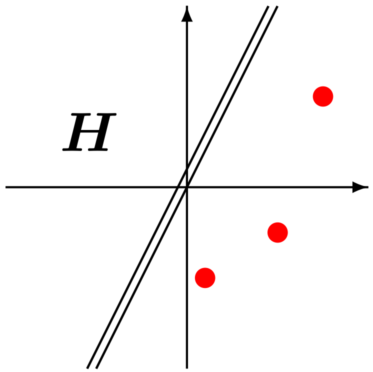
solvable case (OCS) $\Longleftrightarrow$ discrete convex

J3.
Polynomial with Half-Plane Property

Polynomial with Half-Plane Property

$P(z_1, \ldots, z_n) = \sum_{\alpha} c_\alpha z^\alpha$: polynomial

$\mathrm{supp}(P) = \{\alpha \mid c_\alpha \neq 0\} \quad \subseteq \mathbb{Z}^n$



H: open half-plane of $\mathbb{C}$ $\quad (0 \in H)$

P: HPP-polynomial $\Leftrightarrow$ $\exists H$ s.t. $P(z) \neq 0$ $(\forall z \in H)$

Thm (Choe-Oxley-Sokal-Wagner 04)

P: homog. multiaffine HPP $\Longrightarrow$ $\mathrm{supp}(P)$: matroid base

Thm (general) (Brändén 07)

P: HPP $\Longrightarrow$ $\mathrm{supp}(P)$: jump system (possibly non-even)

M-convexity from HPP-Polynomial

$P(z_1, \ldots, z_n) = \sum_{\alpha} c_\alpha(t) z^\alpha$: polynomial

$c_\alpha(t)$: Puiseux (fractional power) series in t

$\nu(c_\alpha(t))$: leading exponent $\quad c_\alpha(t) = b\ t^{-\nu} + \cdots$

$\alpha \mapsto \nu(c_\alpha(t)) \quad \Longrightarrow \mathrm{trop}(P) : \mathrm{supp}(P) \to \mathrm{R}$

Thm (Brändén 10)

Assume: $\mathrm{supp}(P)$ is a const-parity jump system

P: HPP $\Longrightarrow \mathrm{trop}(P)$: M-concave function

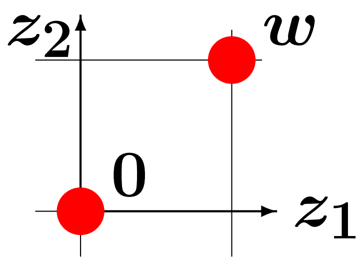
$\mathrm{supp}(P) = \{\alpha \mid c_\alpha(t) \neq 0\}$

P: HPP-polynomial $\Leftrightarrow \exists H$ s.t. $P(z) \neq 0\ (\forall z \in H)$

Examples of HPP-Polynomials — Brändén 10

(1) $P(z_1, z_2) = 1 + t^w z_1 z_2$

H = right half-plane



(2) degree sequence; $w(ij)$: weight of edge ij

$$P(z_1, \ldots, z_n) = \Pi_{ij \in E} \left(1 + t^{w(ij)} z_i z_j\right)$$

$\Rightarrow \mathrm{trop}P(\alpha)$ = max weight of H with $\deg_H = \alpha$

$\rightarrow$ alternative proof to M-concavity

(3) positive-semidefinite matrices $A_1(t), \ldots, A_n(t)$

$$P(z_1, \ldots, z_n) = \det\left(z_1 A_1(t) + \cdots + z_n A_n(t)\right)$$

$\rightarrow$ connection to “hive” (n=3)

J4.
Sum and Convolution

Sum/Convolution Operations

Minkowski Sum (discrete system)

$$J_1 + J_2 = \{\ x_1 + x_2 \mid x_1 \in J_1, x_2 \in J_2\ \}$$

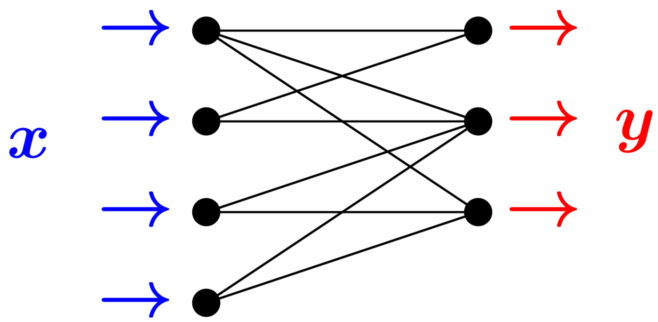
matroid	Rado (42) , Edmonds (68)
base polyhedron	McDiarmid (75)
Δ-matroid	Bouchet (89)
jump system	Bouchet–Cunningham (95)

Convolution (discrete function)

$$(f_1 \Box f_2)(x) = \inf\{f_1(x_1) + f_2(x_2) \mid x_1 + x_2 = x\}$$

valuated matroid	Murota (96)
M-convex (base)	Murota (96)
valuated Δ-matroid	$\downarrow$
M-convex (jump)	Kobayashi-Murota-Tanaka (07)

Transformation by Graph/Network

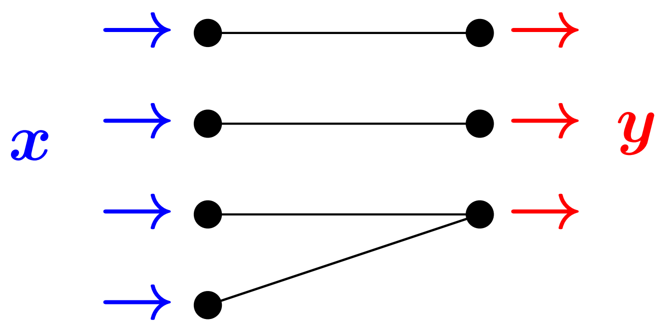


$$\min\{f(x) \mid x \longleftrightarrow y\} := g(y)$$

Thm: (Kobayashi-Murota-Tanaka 07)

f : M-convex $\Longrightarrow$ g: M-convex

Proof by
elementary construction:



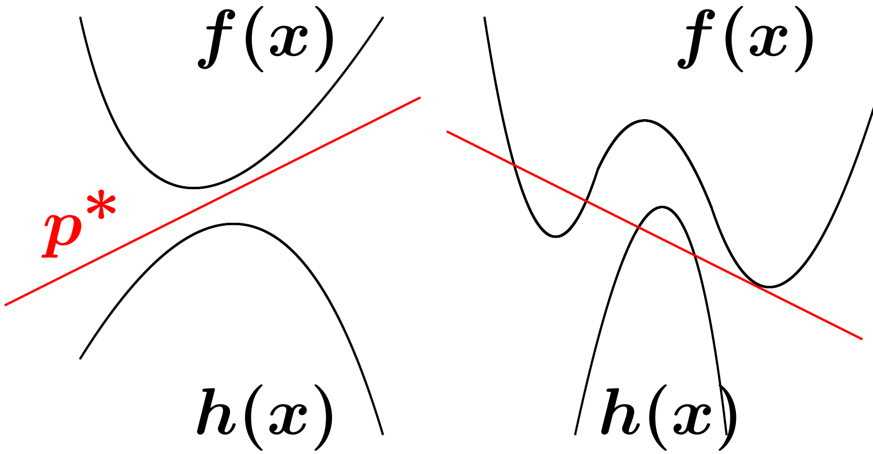
J5.

Duality

Discrete Separation Theorem

$f : \mathbb{Z}^n \to \mathbb{R}$ "convex"

$h : \mathbb{Z}^n \to \mathbb{R}$ "concave"



- $f(x) \geq h(x) \quad (\forall x \in \mathbb{Z}^n) \Rightarrow \exists\ \alpha^* \in \mathbb{R}, \quad \exists\ p^* \in \mathbb{R}^n$:

$$f(x) \geq \alpha^* + \langle p^*, x \rangle \geq h(x) \qquad (x \in \mathbb{Z}^n)$$

- f, h: **integer-valued** $\Rightarrow \alpha^* \in \mathbb{Z}, \quad p^* \in \mathbb{Z}^n$

Frank's Discrete Separation

(Frank 82)

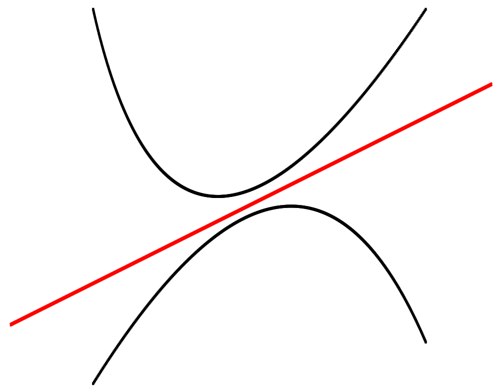
$\rho : 2^V \to \mathrm{R}$: **submodular** $\quad (\rho(\emptyset) = 0)$

$\mu : 2^V \to \mathrm{R}$: **supermodular** $\quad (\mu(\emptyset) = 0)$

- $\rho(X) \geq \mu(X) \quad (\forall X \subseteq V) \Rightarrow \exists\ x^* \in \mathrm{R}^V$:

$$\rho(X) \geq x^*(X) \geq \mu(X) \qquad (\forall X \subseteq V)$$

- ρ, μ: **integer-valued** $\Rightarrow x^* \in \mathrm{Z}^V$



Equivalent to Edmonds' polymatroid intersection

Discrete Separation Theorems

(Murota 96/98)

M-separation Thm: M$^\natural$-convex fn

$\supset$ Weight splitting for weighted matroid intersection

(Iri-Tomizawa 76, Frank 81)

(linear fn, indicator fn $=$ M$^\natural$-convex fn)

L-separation Thm: L$^\natural$-convex fn

$\supset$ Discrete separation for submod. set function

(Frank 82)

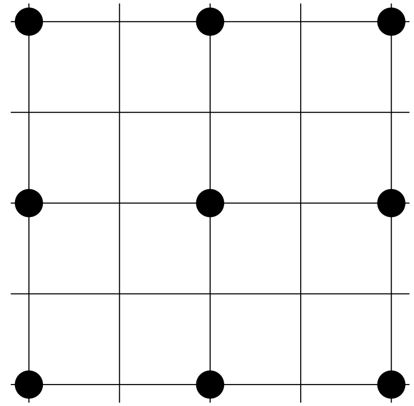
(submod. set fn $=$ L$^\natural$-convex fn on 0–1 vectors)

No Separation Theorem for Jump-M

Even real-separation does not hold

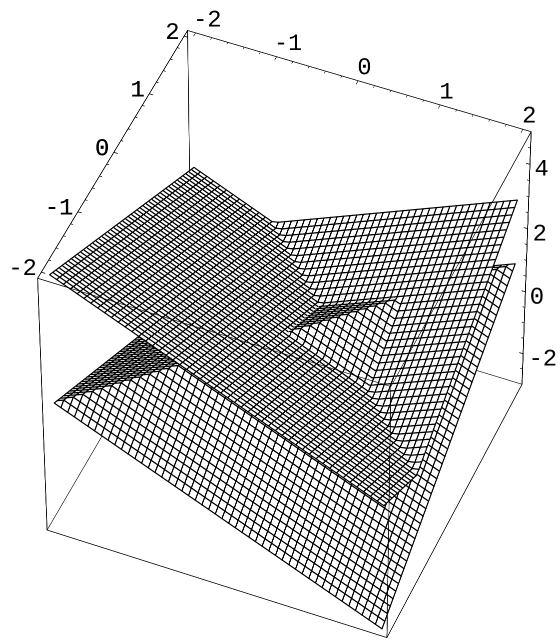
$J = \{(x,y) \mid x,y:$ **even** $\}$

const-parity jump



$f(x,y) = |x+y-2|$ **M-convex**

$h(x,y) = 2 - |x-y|$ **M-concave**



- $f(x,y) \geq h(x,y)$ $(\forall (x,y) \in J)$ **true**
- **No** $\alpha^* \in \mathrm{R}$, $p^* \in \mathrm{R}^2$ **satisfies**

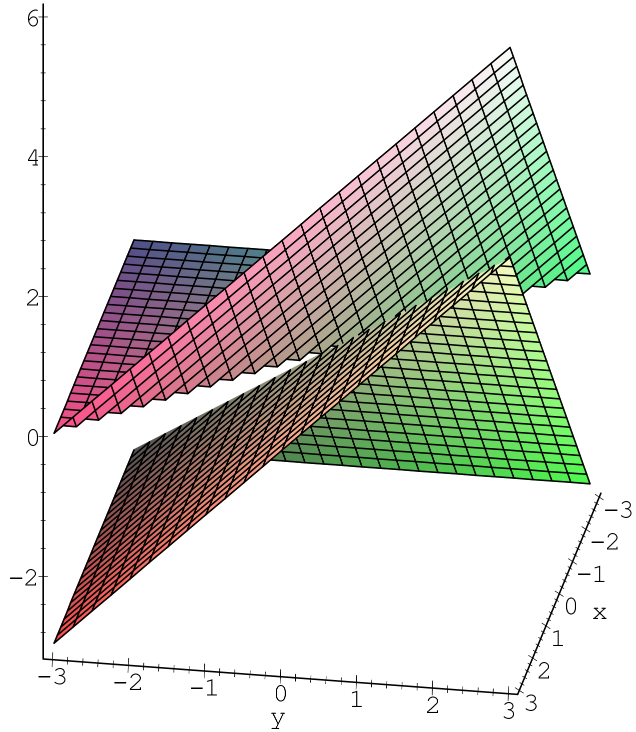
$$f(x) \geq \alpha^* + \langle p^*, x \rangle \geq h(x)$$

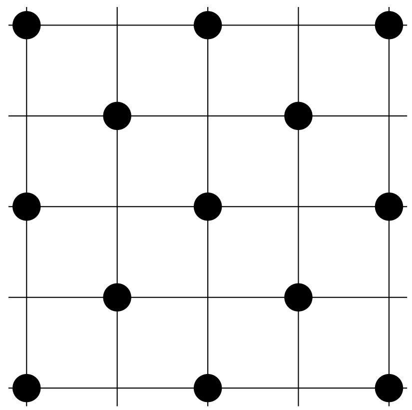
No Integral Separation for Jump-M

$J = \{(x, y) \mid x \equiv y \text{ mod } 2\}$: const-parity jump

$f(x, y) = \max(0, x + y)$ M-convex

$h(x, y) = \min(x, y)$ M-concave





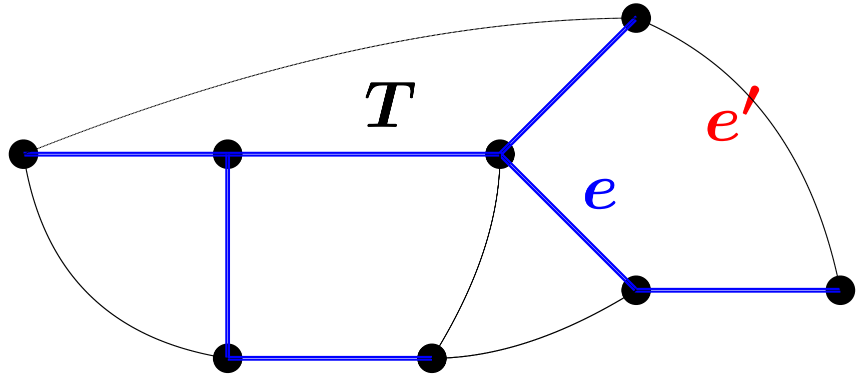
separable, but nonintegral

$p^* = (1/2, 1/2)$, $\alpha^* = 0$ unique separating plane

J6.

M-convex Minimization

Min Spanning Tree Problem



length $d : E \to \mathrm{R}$

total length of T

$$\tilde{d}(T) = \sum_{e \in T} d(e)$$

Thm

T: MST $\iff \tilde{d}(T) \leq \tilde{d}(T - e + e')$

$\iff d(e) \leq d(e')$ if $T - e + e'$ is tree

Algorithm Kruskal's, Kalaba's

DCA view

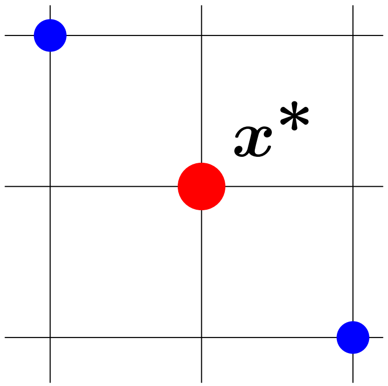
- linear optimization on an M-convex set
- M-optimality: $f(x^*) \leq f(x^* - e_i + e_j)$

Local vs Global Opt (M-base)

Thm: $f : \mathbf{Z}^n \to \mathbf{R}$ M-convex (Murota 96)

x^*: global min

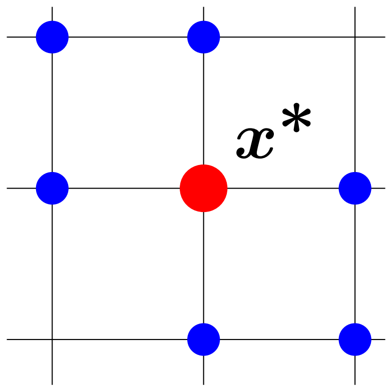
$\iff$ local min $f(x^*) \leq f(x^* - e_i + e_j)$ $(\forall i \neq j)$



Ex: $x^* + (0, 1, 0, 0, -1, 0, 0, 0)$

Can check with n^2 fn evals

For M$^\natural$-convex fn $\Rightarrow$



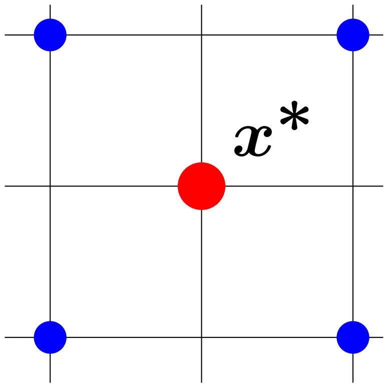
Local vs Global Opt (M-jump)

Thm: (Murota 06)

$f : J \to \mathbb{R}$ M-convex on const-parity jump

x^*: global min

$\iff$ local min $\quad f(x^*) \le f(x^* \pm e_i \pm e_j) \quad (\forall i, j)$



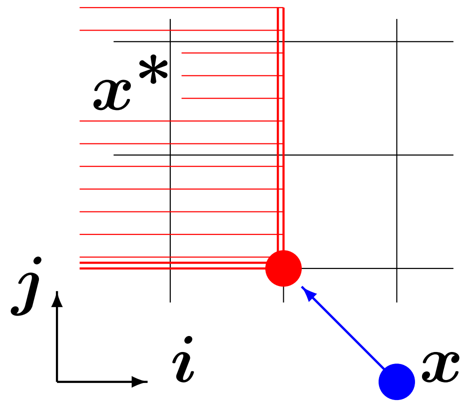
Ex: $x^* + (0, \pm 1, 0, 0, \pm 1, 0, 0, 0)$

Can check with n^2 fn evals

Minimizer Cut Theorems

⇒ Domain reduction algorithm

⇒ Descent algorithm



Thm (M-base) $x \notin \arg\min f$ (Shioura 98)

For (i, j) that minimizes $f(x - e_i + e_j)$,

$\exists$ minimizer x^* with $x^*_i \leq x_i - 1, \; x^*_j \geq x_j + 1$

Thm (M-jump) $x \notin \arg\min f$ (Murota-Tanaka 06)

For $(i, j; \sigma_i, \sigma_j)$ that minimizes $f(x \pm e_i \pm e_j)$,

$\pm \; \pm$

$\exists$ minimizer x^* with $\sigma_i(x^*_i - x_i) \geq 1, \; \sigma_j(x^*_j - x_j) \geq 1$

Sum-Constrained Minimization

Problem [k]: Min. $f(x)$ s.t. $x \in J_k$

J: const-parity jump, $J_k = \{x \in J \mid \Sigma x_i = k\}$

Thm: (Murota 06)

- global min $\iff$ local min wrt $\| \|_1 \leq 4$
- convex optimal values:

$$f_{\text{opt}}(k-1) + f_{\text{opt}}(k+1) \geq 2\, f_{\text{opt}}(k)$$

- nested minimizers:

$$\forall x_{\text{opt}}(k), \ \exists x_{\text{opt}}(k-1), \ \exists x_{\text{opt}}(k+1):$$

$$x_{\text{opt}}(k-1) \leq x_{\text{opt}}(k) \leq x_{\text{opt}}(k+1)$$

Proof: Local $\| \|_1 \leq 4 \Rightarrow$ Global opt

Problem [k]: Min. $f(x)$ s.t. $x \in J$, $\sum x_i = k$

• You do not have to read this slide •

By way of contradiction, assume
$f(x) > f(y)$ **for** $y \in J_k$ **with** $||y - x||_1 \to \min$
If you are still reading, you are already in contradiction :-)

Claim: $\forall i \in \mathrm{supp}^+(y-x)$, $\forall j \in \mathrm{supp}^-(y-x)$:
$f(x) + f(y) < 2f(x) \leq f(x + e_i - e_j) + f(y - e_i + e_j)$

(M-EXC) for (x, y) & Claim,
$\exists\ i_1 \in \mathrm{supp}^+(y-x)$, $i_2 \in \mathrm{supp}^+(y - x - e_{i_1})$,
$j_1 \in \mathrm{supp}^-(y-x)$, $j_2 \in \mathrm{supp}^-(y - x - e_{j_1})$:

$$f(x) + f(y) \geq f(x + e_{i_1} + e_{i_2}) + f(y - e_{i_1} - e_{i_2}) \quad (1)$$
$$f(x) + f(y) \geq f(x - e_{j_1} - e_{j_2}) + f(y + e_{j_1} + e_{j_2}) \quad (2)$$

(continued)

(M-EXC) for $(x + e_{i_1} + e_{i_2}, x - e_{j_1} - e_{j_2})$ **& local opt,**

$$
\begin{aligned}
& f(x + e_{i_1} + e_{i_2}) + f(x - e_{j_1} - e_{j_2}) \\
& \geq \min[f(x + e_{i_1} - e_{j_1}) + f(x + e_{i_2} - e_{j_2}), \\
& \qquad f(x + e_{i_1} - e_{j_2}) + f(x + e_{i_2} - e_{j_1}), \\
& \qquad f(x) + f(x + e_{i_1} + e_{i_2} - e_{j_1} - e_{j_2})] \\
& \geq 2f(x) \qquad (3)
\end{aligned}
$$

(M-EXC) for $(y - e_{i_1} - e_{i_2}, y + e_{j_1} + e_{j_2})$**,**

$$
\begin{aligned}
& f(y - e_{i_1} - e_{i_2}) + f(y + e_{j_1} + e_{j_2}) \\
& \geq \min[f(y - e_{i_1} + e_{j_1}) + f(y - e_{i_2} + e_{j_2}), \\
& \qquad f(y - e_{i_1} + e_{j_2}) + f(y - e_{i_2} + e_{j_1}), \\
& \qquad f(y) + f(y - e_{i_1} - e_{i_2} + e_{j_1} + e_{j_2})] \\
& \geq f(x) + f(y) \qquad (4)
\end{aligned}
$$

---

(1)+(2)+(3)+(4) = contradiction **Q.E.D.**

Optimality Criteria

Problem [$*$]: Min. $f(x)$ s.t. $x \in J$

Problem [k]: Min. $f(x)$ s.t. $x \in J$, $\sum x_i = k$

	Problem [$*$] Local: $\|\|\cdot\|\|_1 \leq 2$	Problem [k] Local: $\|\|\cdot\|\|_1 \leq 4$
jump: M-convex	Murota (06)	Murota (06)
jump: separ. conv	Ando-Fujishige-Naitoh (95)	Apollonio-Sebő (04) degree sequence
valuated Δ-matroid	Dress-Wenzel (91)	Murota (96)
valuated matroid	Dress-Wenzel (90)	——— ———
polymatroid: M/M$^\natural$-conv	Murota (96) Murota-Shioura (99)	Murota-Shioura (99)

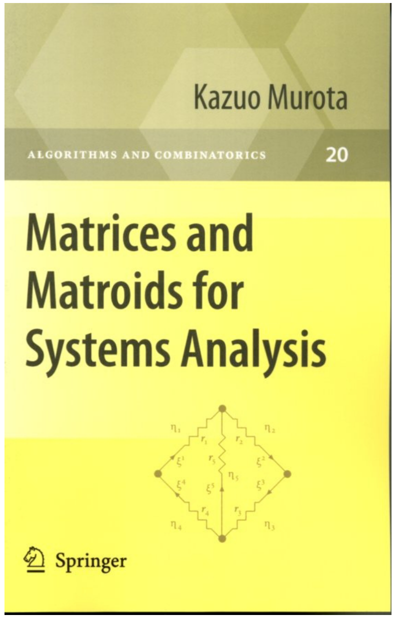
Books

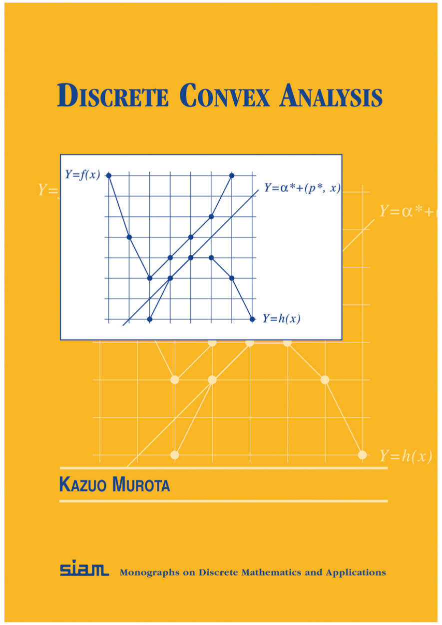
Murota: Matrices and Matroids for Systems Analysis, Springer, 2000/2010 (Chap.5)

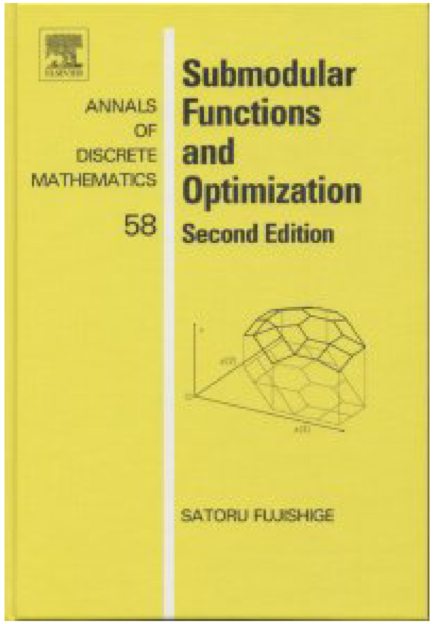
valuated matroid intersection algorithm

Murota: Discrete Convex Analysis, SIAM, 2003

Fujishige: Submodular Functions and Optimization, 2nd ed., Elsevier, 2005 (Chap. VII)







END