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Convexity and Optimization: Theory and Applications

DC Programming in Discrete Convex Analysis

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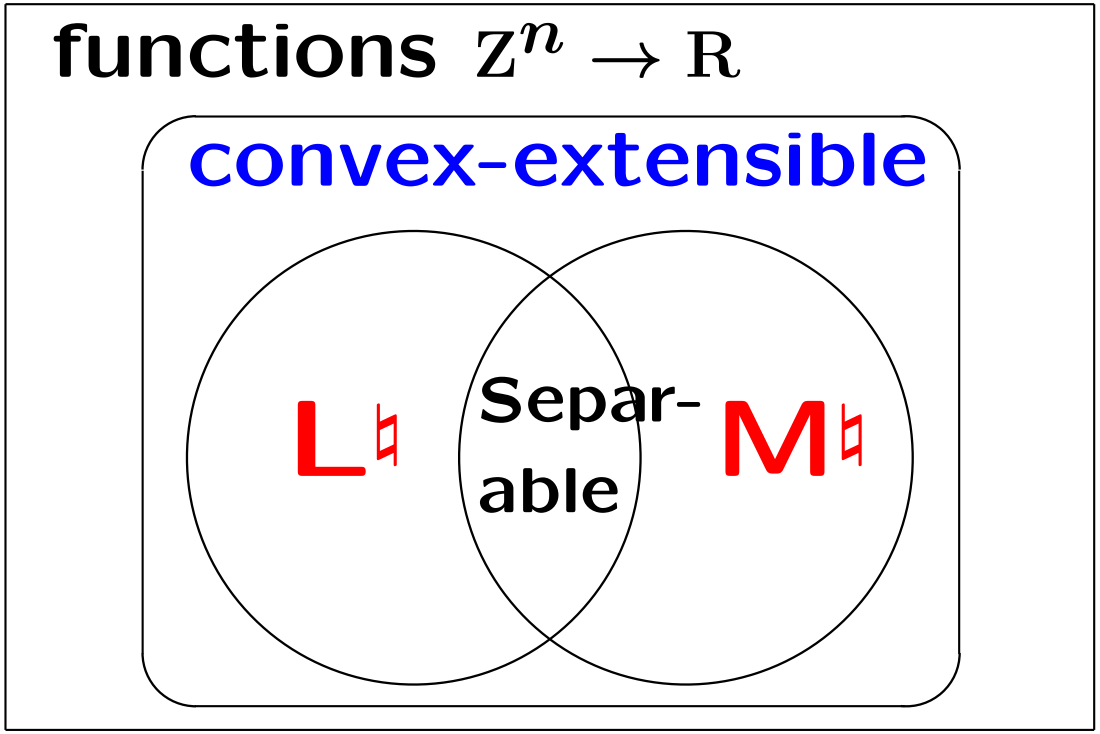
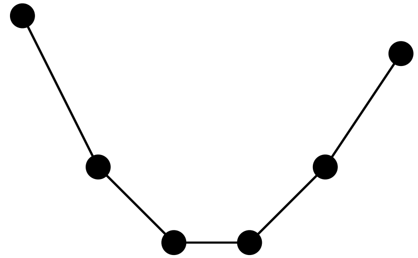
Discrete Convex Functions



$f : \mathbb{Z}^n \to \mathbb{R}$



f is convex-extensible

$\Leftrightarrow$ $\exists$ convex $\overline{f}$:

$f(x) = \overline{f}(x)$

f is separable $\Leftrightarrow$ $f(x) = \varphi_1(x_1) + \cdots + \varphi_n(x_n)$

Discrete DC Program

DC = Difference of Convex

$$\min_{x \in \mathbf{Z}^n} \{g(x) - h(x)\}$$

g, h: "convex" $(\mathrm{dom}\, g \subseteq \mathrm{dom}\, h)$

convexity: M$^\natural$–M$^\natural$, M$^\natural$–L$^\natural$, L$^\natural$–M$^\natural$, L$^\natural$–L$^\natural$

(Examples)

Submod. max. under matroid constraint: M$^\natural$–L$^\natural$

Subm-superm proc. (Narasimhan-Bilmes): L$^\natural$–L$^\natural$

DC Algorithm (Pham Dinh Tao 1985(ca.))

$$\min\{g(x) - h(x)\} \implies \min\{g(x) - \langle p, x\rangle\}$$

subgradient $p \in \partial h(x)$

Algorithm 1 DC algorithm

Let $x^{(1)}$ be an initial solution
for $k = 1, 2, \ldots$ **do**
(Dual phase) Pick $p^{(k)} \in \partial h(x^{(k)})$
(Primal phase) Pick $x^{(k+1)} \in \operatorname{argmin}(g - p^{(k)})$
if $(g - p^{(k)})(x^{(k)}) = (g - p^{(k)})(x^{(k+1)})$ **then**
Return $x^{(k)}$
end if
end for

- **concave-convex proc (Yuille–Rangarajan 03)**
- **submod-supermod proc (Narasimhan–Bilmes 05)**

cf. supermod-submod proc, mod-mod proc
(Iyer–Bilmes 12; Iyer–Jegelka–Bilmes 13)

Subdifferentiability & Biconjugacy

$$\partial f(x) = \{p \mid f(y) - f(x) \geq \langle p, y - x \rangle \ \ (\forall y)\}$$

$$f^{\bullet}(p) = \sup_{x} \ \{\langle p, x \rangle - f(x)\}$$

$$p \in \arg\min_{q} \{f^{\bullet}(q) - \langle q, x \rangle\} \iff p \in \partial f(x)$$

$$\Updownarrow$$

$$x \in \arg\min_{x} \{f(y) - \langle p, y \rangle\} \iff x \in \partial f^{\bullet}(p)$$

biconjugacy: $f^{\bullet\bullet} = f$

Integral Subgradients & Biconjugacy

(Discrete life is not easy)

$f : \mathrm{Z}^n \to \overline{\mathrm{Z}}$ $\partial_{\mathrm{Z}} f(x) \neq \emptyset$? $f^{\bullet\bullet} = f$?

Example: $D = \{(0,0,0), \pm(1,1,0), \pm(0,1,1), \pm(1,0,1)\}$

$$f(x_1, x_2, x_3) = \begin{cases} (x_1 + x_2 + x_3)/2, & x \in D, \\ +\infty, & \text{o.w.} \end{cases}$$

D is "convex": $\mathrm{conv}(D) \cap \mathrm{Z}^n = D$

$\partial f_{\mathrm{R}}(0) = \{(1/2, 1/2, 1/2)\}$ $\partial_{\mathrm{Z}} f(0) = \emptyset$

$$f^{\bullet\bullet}(0) = -\inf_{p \in \mathbb{Z}^3} \max\{0, |p_1 + p_2 - 1|, |p_2 + p_3 - 1|, |p_3 + p_1 - 1|\}$$

$f^{\bullet\bullet}(0) = -1 \neq 0 = f(0)$

Biconjugacy in Matroids (promising clue)

Biconjugacy

Independent sets $\mathcal{I}$		Rank function ρ
Exchange axiom	$\Longleftrightarrow$	Submodularity
(vertex)	$\Longleftrightarrow$	(face)

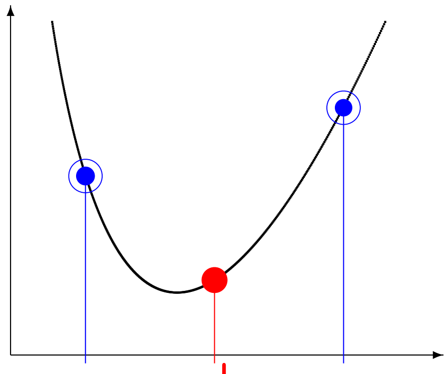
Subgradients

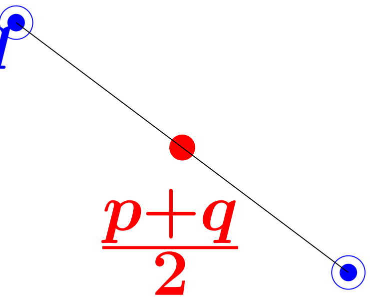
$$\partial\rho(\emptyset) = \text{Independence polyhedron}$$

L-convex and M-convex Functions

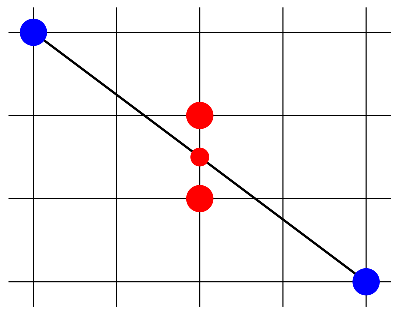
L$^\natural$-convexity from Mid-pt-convexity

(Favati-Tardella 1990, Murota 1998, Fujishige–Murota 2000)





$\Longrightarrow$



Mid-point convex ($g : \mathrm{R}^n \to \mathrm{R}$):

$$g(p) + g(q) \ \geq \ 2g\left(\frac{p+q}{2}\right)$$

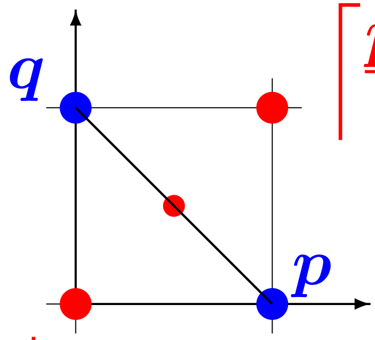
$\Longrightarrow$ **Discrete mid-point convex** ($g : \mathrm{Z}^n \to \mathrm{R}$)

$$g(p) + g(q) \ \geq \ g\left(\left\lceil\frac{p+q}{2}\right\rceil\right) + g\left(\left\lfloor\frac{p+q}{2}\right\rfloor\right)$$

L$^\natural$-convex function (L = Lattice)

Mid-pt Convexity for 01-Vectors

For $p, q \in \{0,1\}^n$**:**



$$\left\lceil \frac{p+q}{2} \right\rceil = p \vee q$$

$$\left\lfloor \frac{p+q}{2} \right\rfloor = p \wedge q$$

Discrete mid-pt convexity:

$$g(p) + g(q) \;\geq\; g\left(\left\lceil \frac{p+q}{2} \right\rceil\right) + g\left(\left\lfloor \frac{p+q}{2} \right\rfloor\right)$$

$\iff$ **Submodularity:**

$$g(p) + g(q) \;\geq\; g\left(p \vee q\right) + g\left(p \wedge q\right)$$

L$^\natural$-convexity from Submodularity

—original definition (Murota 1998) —

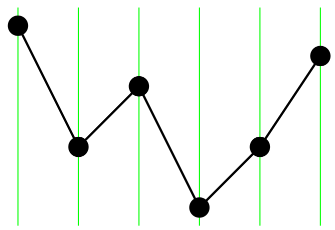
Def: $g : \mathbf{Z}^n \to \mathbf{R}$ L$^\natural$-convex $\iff$

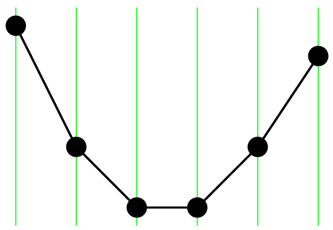
$\tilde{g}(p_0, p) = g(p - p_0 \mathbf{1})$ is submodular in (p_0, p)

$\tilde{g} : \mathbf{Z}^{n+1} \to \mathbf{R}$, $\qquad \mathbf{1} = (1, 1, \ldots, 1, 1)$

Rem: L♮-convex vs Submodular

$n = 1$





Fact 1: Any $g : \mathbb{Z} \to \mathbb{R}$ is submodular

Fact 2: Function $g : \mathbb{Z} \to \mathbb{R}$ is L♮-convex

$\iff g(p-1) + g(p+1) \geq 2g(p)$ for all $p \in \mathbb{Z}$

L$^\natural$-convex Function: Examples

Quadratic: $g(p) = \sum_i \sum_j a_{ij} p_i p_j$ is L$^\natural$-convex

$$\Leftrightarrow a_{ij} \leq 0 \quad (i \neq j), \qquad \sum_j a_{ij} \geq 0 \ (\forall i)$$

Separable convex: For univariate convex ψ_i and ψ_{ij}

$$g(p) = \sum_i \psi_i(p_i) + \sum_{i \neq j} \psi_{ij}(p_i - p_j)$$

Range: $g(p) = \max\{p_1, p_2, \ldots, p_n\} - \min\{p_1, p_2, \ldots, p_n\}$

Submodular set function: $\rho : 2^V \to \overline{\mathbf{R}}$

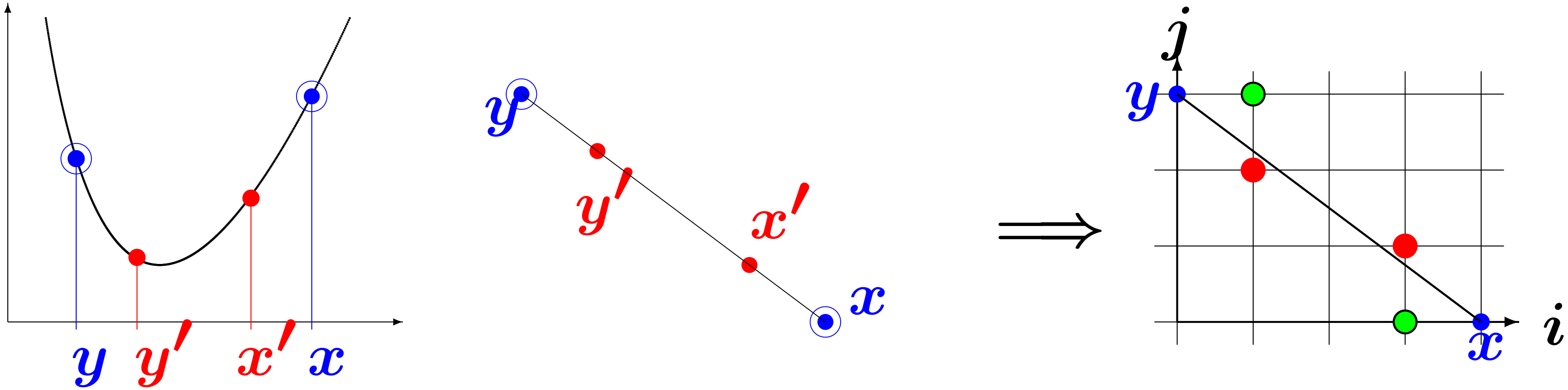
$\Leftrightarrow \rho(X) = g(\chi_X)$ for some L$^\natural$-convex g

Multimodular: $h : \mathbf{Z}^n \to \overline{\mathbf{R}}$ is multimodular $\Leftrightarrow$

$h(p) = g(p_1, p_1 + p_2, \ldots, p_1 + \cdots + p_n)$ for L$^\natural$-convex g

M♮-convexity from Equi-dist-convexity

(Murota 1996, Murota–Shioura 1999)



Equi-distance convex ($f: \mathrm{R}^n \to \mathrm{R}$):

$$f(x) + f(y) \geq f(x - \alpha(x-y)) + f(y + \alpha(x-y))$$

$\Longrightarrow$ **Exchange** ($f: \mathrm{Z}^n \to \mathrm{R}$) $\quad \forall x, y, \quad \forall i: x_i > y_i$

$$f(x) + f(y) \geq \min \Big[f(x - e_i) + f(y + e_i), \\ \min_{x_j < y_j} \{ f(x - e_i + e_j) + f(y + e_i - e_j) \} \Big]$$

M♮-convex function (M = Matroid)

M$^\natural$-convex Function: Examples

Quadratic: $f(x) = \sum_i \sum_j a_{ij} x_i x_j$ is M$^\natural$-convex

$$\Leftrightarrow a_{ij} \geq 0, \quad a_{ij} \geq \min(a_{ik}, a_{jk}) \ (\forall k \notin \{i, j\})$$

Min value: $f(X) = \min\{a_i \mid i \in X\}$ [unit preference]

Matroid rank: $f(X) = -$ rank of X

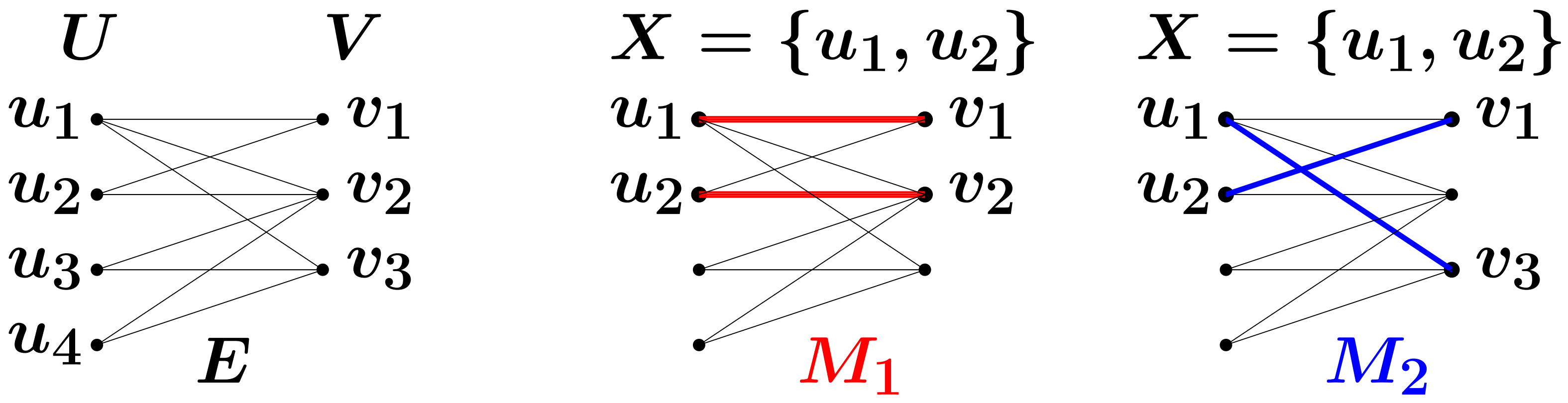
Cardinality convex: $f(X) = \varphi(|X|)$ (φ: convex)

Separable convex: $f(x) = \sum_i \varphi_i(x_i)$ (φ_i: convex)

Laminar convex: $f(x) = \sum_A \varphi_A(x(A))$ (φ_A: convex)

$\{A, B, \ldots.\}$: laminar $\Leftrightarrow A \cap B = \emptyset$ or $A \subseteq B$ or $A \supseteq B$

Matching / Assignment



Max weight for $X \subseteq U$ **(w: given weight)**

$$f(X) = \max\{ \sum_{e \in M} w(e) \mid M\text{: matching, } U \cap \partial M = X \}$$

Max-weight func f is M$^\natural$-concave (Murota 1996)

- Proof by augmenting path
- Extension to min-cost network flow

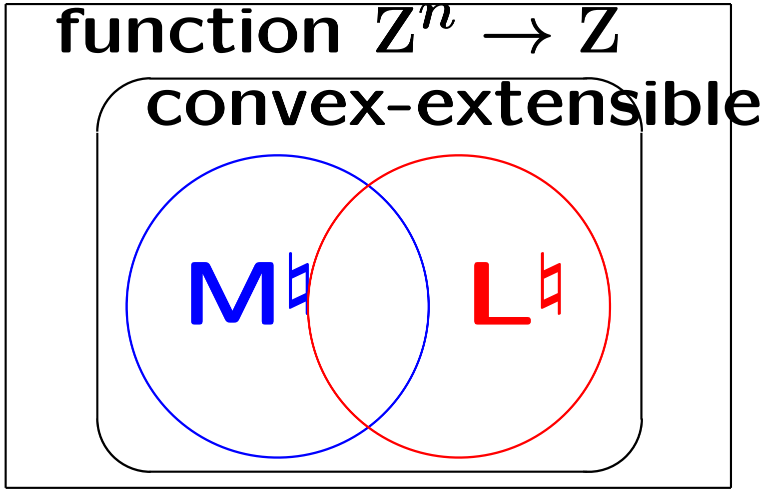
M-L Conjugacy Theorem

Integer-valued discrete fn $f : \mathbf{Z}^n \to \overline{\mathbf{Z}}$

Legendre transform: $f^\bullet(p) = \sup_{x \in \mathbf{Z}^n} [\langle p, x \rangle - f(x)]$

M♮-convex and L♮-convex are conjugate

$f \mapsto f^\bullet = g \mapsto g^\bullet = f$ (Murota 1998)



biconjugacy

$f^{\bullet\bullet} = f$

Discrete Toland-Singer Duality

$f^\bullet(p) = \sup\{\langle p, x\rangle - f(x) \mid x \in \mathbf{Z}^n\}$

$h : \mathbf{Z}^n \to \mathbf{Z}$: M$^\natural$-convex or L$^\natural$-convex (g: any fn)

Toland-Singer Duality (Maehara-Murota 13)

$$\inf_{x\in\mathbf{Z}^n}\{g(x) - h(x)\} = \inf_{p\in\mathbf{Z}^n}\{h^\bullet(p) - g^\bullet(p)\}$$

(Proof) Integral biconjugacy: $h^{\bullet\bullet} = h$.

$$\inf_x\{g(x) - h(x)\} = \inf_x\{g(x) - h^{\bullet\bullet}(x)\}$$
$$= \inf_x\{g(x) - \sup_p\{\langle p, x\rangle - h^\bullet(p)\}\}$$
$$= \inf_x \inf_p\{g(x) - \langle p, x\rangle + h^\bullet(p)\}$$
$$= \inf_p\{h^\bullet(p) - \sup_x\{\langle p, x\rangle - g(x)\}\} = \inf_p\{h^\bullet(p) - g^\bullet(p)\}.$$

Toland-Singer vs Fenchel

$$f^\bullet(p) = \sup\{\langle p, x\rangle - f(x) \mid x \in \mathbf{Z}^n\}$$

Toland-Singer Duality (Maehara-Murota 13)

$$\inf_{x \in \mathbf{Z}^n}\{g(x) - h(x)\} = \inf_{p \in \mathbf{Z}^n}\{h^\bullet(p) - g^\bullet(p)\}$$

(g, h)= (any, M$^\natural$), (any, L$^\natural$)

Fenchel Duality (Murota 1996, 1998)

$$\inf_{x \in \mathbf{Z}^n}\{g(x) + h(x)\} = -\inf_{p \in \mathbf{Z}^n}\{h^\bullet(p) + g^\bullet(-p)\}$$

(g, h)= (M$^\natural$, M$^\natural$), (L$^\natural$, L$^\natural$)

containing: Edmonds's matroid intersection
Frank's weight splitting for wtd matroid intersection
Fujishige's Fenchel duality for submod set functions

Submodularity and Convexity

—1980's vs 2000's —

Submodularity & Convexity in 1980's

$$\rho(X) + \rho(Y) \geq \rho(X \cup Y) + \rho(X \cap Y)$$

- min/max algorithms

(Grötschel–Lovász–Schrijver/ Jensen–Korte, Lovász)

min $\Rightarrow$ polynomial, max $\Rightarrow$ NP-hard

- Convex extension (Lovász)

set fn is submod $\Leftrightarrow$ Lovász ext is convex

- Duality theorems (Edmonds, Frank, Fujishige)

discrete separation, Fenchel min-max

Submodular set function
= Convexity + Discreteness

But ...

decreasing
marginal return $\longleftrightarrow$ concave/submodular

This means ...

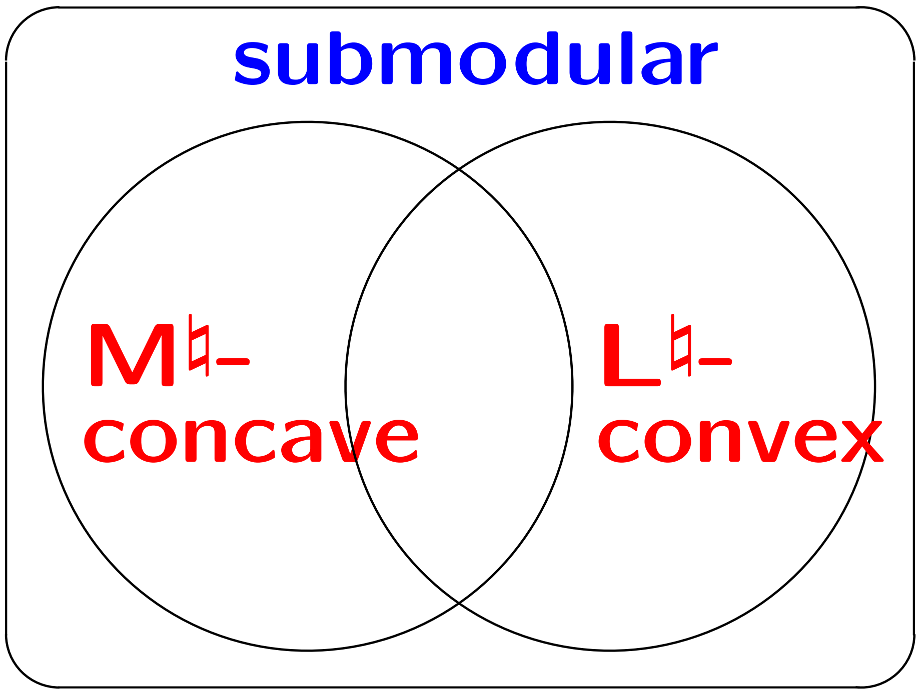
Submodular $\approx$ Concave

We know ...

$\rho(X) = \varphi(|X|)$ (φ: concave) is submodular

Submodularity & Convexity in DCA

- M$^\natural$-concave function is submodular
- L$^\natural$-convex function is submodular



- Sum of M$^\natural$-concave fns is submodular
- Sum of L$^\natural$-convex fns is L$^\natural$-conv (subm)

Representability of DC fns

Complexity of DC programs

Discrete DC Functions: Representability

Discrete DC = DDC = Difference of Discrete Convex

$L^\natural$–$L^\natural$: (almost) all functions

$L^\natural$–$M^\natural$: $\subseteq$ submodular

$M^\natural$–$L^\natural$: $\subseteq$ supermodular

$M^\natural$–$M^\natural$: ? – Not every function is $M^\natural$–$M^\natural$

Example of Non-M♮–M♮ DC Function

$f(x, y, z, w) = xz + xw + yz$ is not M♮–M♮ DC
(No h exists s.t. both $f + h$ and h are M♮-convex)

(Proof) discrete Hessian $H(x) = [H_{ij}(x)]$:

$$H_{ij}(x) = f(x + e_i + e_j) - f(x + e_i) - f(x + e_j) + f(x)$$

Theorem: (Hirai–Murota 04, Murota 07)

f is M♮-convex $\Leftrightarrow$ for every $x \in \mathbb{Z}^n$:

$$H_{ij}(x) \geq 0 \quad H_{ij}(x) \geq \min\{H_{ik}(x), H_{jk}(x)\} \ (i, j \neq k)$$

Both $H_f + H_h$ and H_h must satisfy this condition

$$H_f = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad H_h = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

Complexity of Discrete DC Programs

DC = Difference of Convex

$$\min_{x \in \mathbf{Z}^n} \{g(x) - h(x)\} \qquad g, h: \text{ “convex”}$$

Convexity: M$^\natural$–M$^\natural$, M$^\natural$–L$^\natural$, L$^\natural$–M$^\natural$, L$^\natural$–L$^\natural$

$x \in \mathbf{Z}^n$

$g \backslash h$	M$^\natural$	L$^\natural$
M$^\natural$	NP-hard	NP-hard
L$^\natural$	open	NP-hard

pseudo-poly; submod min (ring family)

$x \in \{0, 1\}^n$

$g \backslash h$	M$^\natural$	L$^\natural$
M$^\natural$	NP-hard	NP-hard
L$^\natural$	P	NP-hard

Kobayashi 14, submod min

NP-hardness of M$^\natural$–M$^\natural$ DC Programs

- **Matroid rank fns are M$^\natural$-concave** (Fujishige 05)
- **NP-hard Problem:** (Kobayashi 14)

Min $f(X)$ s.t. $|X| = m$ for matroid rank fn f

$\Longleftrightarrow$ Min $f(X) + \delta_m(X)$ (δ_m: indicator for $|X| = m$)

$f + \delta_m = \delta_m - (-f) =$ (M$^\natural$-conv) $-$ (M$^\natural$-conv)

Reduction from Max-Clique

For graph G, integer k,

f = rank function of graphic matroid, $m = \binom{k}{2}$

G has k-clique $\Longleftrightarrow$ $\min(f + \delta_m) = k - 1$

Discrete DC Algorithm

Discrete DC Algorithm

$$\min_{x \in \mathbf{Z}^n} \{g(x) - h(x)\} \quad \Longrightarrow \quad \min_{x \in \mathbf{Z}^n} \{g(x) - \langle p, x \rangle\}$$

integral subgradient $p \in \partial h(x)$

Algorithm 2 Discrete DC algorithm

Let $x^{(1)}$ be an initial solution
for $k = 1, 2, \ldots$ **do**
(Dual phase) Pick $p^{(k)} \in \partial h(x^{(k)}) \setminus \partial g(x^{(k)})$
(Primal phase) Pick $x^{(k+1)} \in \mathrm{argmin}\,(g - p^{(k)})$
if $(g - p^{(k)})(x^{(k)}) = (g - p^{(k)})(x^{(k+1)})$ **then**
Return $x^{(k)}$
end if
end for

- **monotone decreasing** $g(x) - h(x)$
- $\partial g(x) \supseteq \partial h(x)$ **guaranteed**
- **local optimality within** $U = \bigcup_{p \in \partial g(x)} \partial h^{\bullet}(p)$

Optimality Conditions $\min_{x \in \mathbf{Z}^n} \{g(x) - h(x)\}$

x: global opt $\iff$ $\partial_\epsilon g(x) \supseteq \partial_\epsilon h(x) \quad (\forall \epsilon \geq 0)$

$$\Downarrow$$

$$\boxed{\partial g(x) \supseteq \partial h(x)}$$

$$\Downarrow$$

x: local opt, i.e., x: minimum in

$$U = \bigcup_{p \in \partial g(x)} \partial h^{\bullet}(p)$$

$$\partial_{\epsilon} f(x) = \{p \mid f(y) - f(x) \geq \langle p, y - x \rangle - \epsilon \quad (\forall y)\}$$

Subgradient of M♮-/L♮-convex Func

$f : \mathbb{Z}^n \to \overline{\mathbb{Z}}$,

integral subgradients:

$$\partial f(x) = \{p \in \mathbb{Z}^n \mid f(y) - f(x) \geq \langle p, y - x \rangle \ \ (\forall y)\}$$

f: M♮-convex $\Rightarrow$ $\partial f(x) \neq \emptyset$: L♮-convex set

$$-p_i \leq f(x - \chi_i) - f(x), \qquad p_j \leq f(x + \chi_j) - f(x),$$

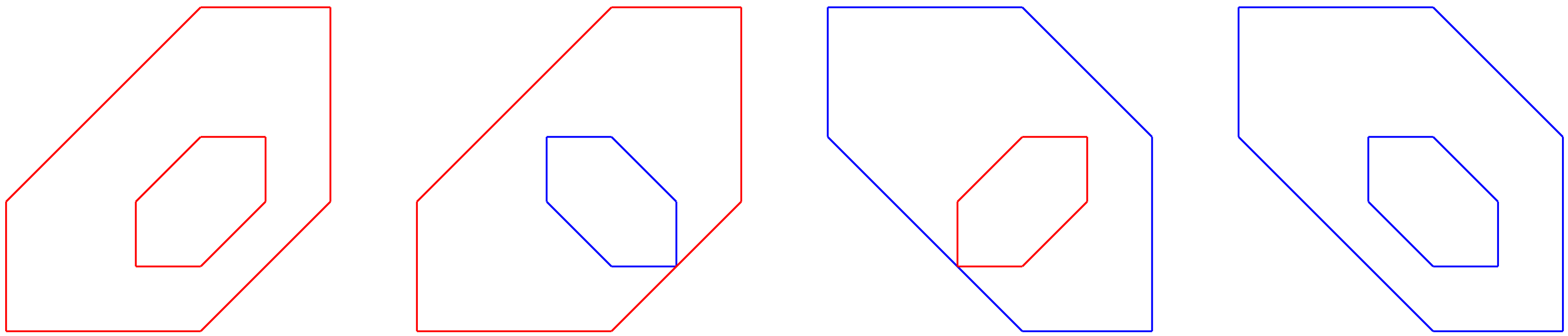
$$p_j - p_i \leq f(x - \chi_i + \chi_j) - f(x) \qquad (\forall i, j)$$

f: L♮-convex $\Rightarrow$ $\partial f(x) \neq \emptyset$: M♮-convex set

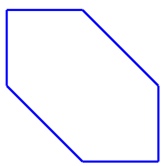
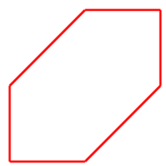
$$f(x) - f(x - \chi_A) \leq \langle p, \chi_A \rangle \leq f(x + \chi_A) - f(x) \quad (\forall A)$$

Testing for Local Opt: $\partial g(x) \supseteq \partial h(x)$

$\mathrm{M}^\natural$–$\mathrm{M}^\natural$	$\mathrm{M}^\natural$–$\mathrm{L}^\natural$	$\mathrm{L}^\natural$–$\mathrm{M}^\natural$	$\mathrm{L}^\natural$–$\mathrm{L}^\natural$



$\mathrm{L}^\natural \supseteq \mathrm{L}^\natural$	$\mathrm{L}^\natural \supseteq \mathrm{M}^\natural$	$\mathrm{M}^\natural \supseteq \mathrm{L}^\natural$	$\mathrm{M}^\natural \supseteq \mathrm{M}^\natural$
poly-time	**poly-time**	**poly-time**	**co-NP-compl.**
$O(n^2)$ ineqs	$O(n^2)$ M-min	submod-min	(McCormick 96)

 g-polymatroid  dual of shortest-path

Larger Conjugate Classes of Functions

(i) sum of two M$^\natural$-convex fns M$^\natural$+M$^\natural$

(ii) convolution of two L$^\natural$-conv fns L$^\natural$ $\Box$ L$^\natural$

Subdifferentiability & Biconjugacy hold

Toland-Singer Duality Thm (extension)

$h : \mathbf{Z}^n \to \mathbf{Z}$ M$^\natural$+M$^\natural$ or L$^\natural$ $\Box$ L$^\natural$ (g: any fn)

$$\inf_{x \in \mathbf{Z}^n} \{g(x) - h(x)\} = \inf_{p \in \mathbf{Z}^n} \{h^\bullet(p) - g^\bullet(p)\}$$

(Preliminary) Computational Results

Max Clique Problem (M♮–M♮ DC)

$G = (V, E)$ graph

$\text{rank}\,(X)$ = maximum cardinality of a forest in $X \subseteq E$

Maximize $\qquad f(X) = \begin{cases} |X| - \text{rank}\,(X), & |X| = \binom{k}{2} \\ -\infty, & \text{otherwise} \end{cases}$

for an edge set incident to small number of vertices

$$f(X) = \frac{k(k-1)}{2} - (k-1) \iff X \text{ is a } k\text{-clique}$$

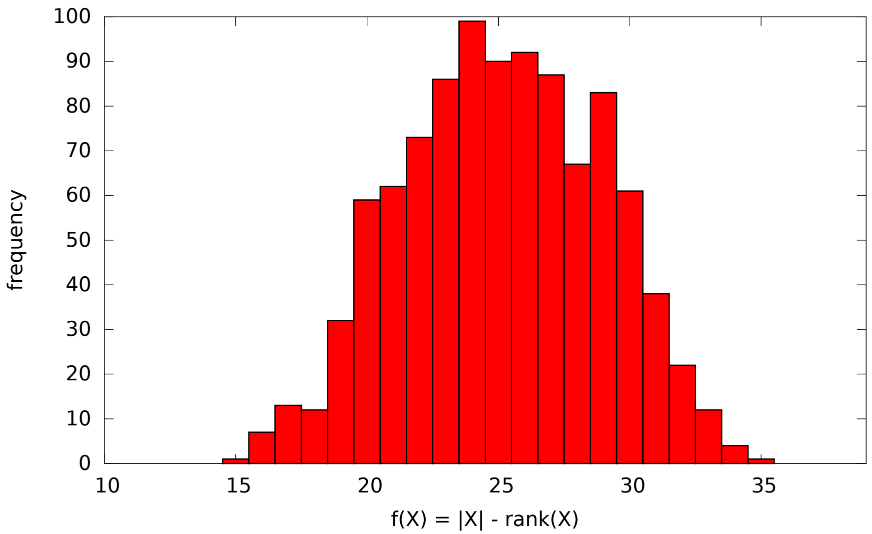
Max Clique: random graph (M♮–M♮ DC)

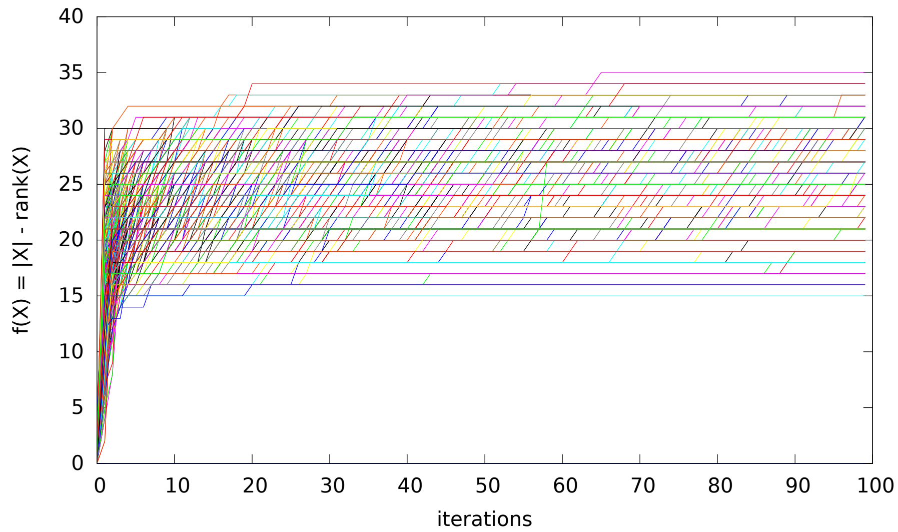
$f(X) = |X| - \mathrm{rank}(X)$ if $|X| = \binom{k}{2}$

Erdős–Rényi graph $n = 100$, $p = 0.1$ (av. degree 10)

+ clique of size $k = 10$

$\max f = 36$, Best $\max f = 35$ in 1000 runs





Dolphins Network (M$^\natural$–M$^\natural$ DC)

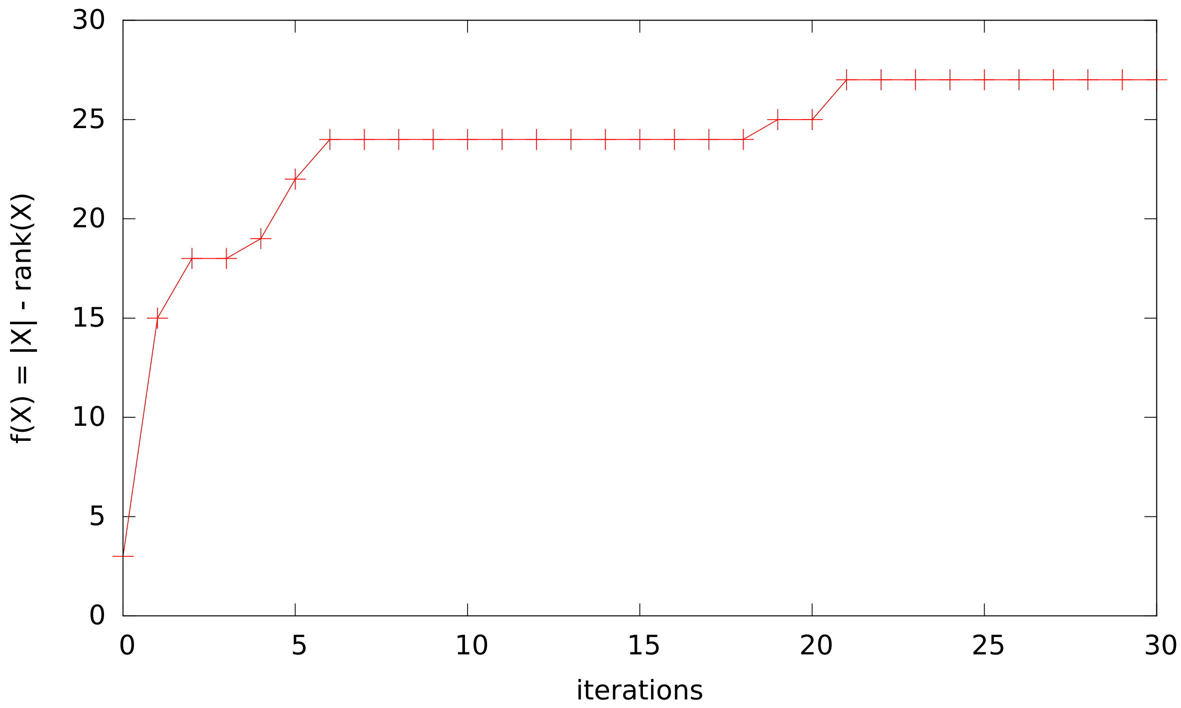
(http://konect.uni-koblenz.de/networks/dolphins)

Representing friendship relation between dolphins

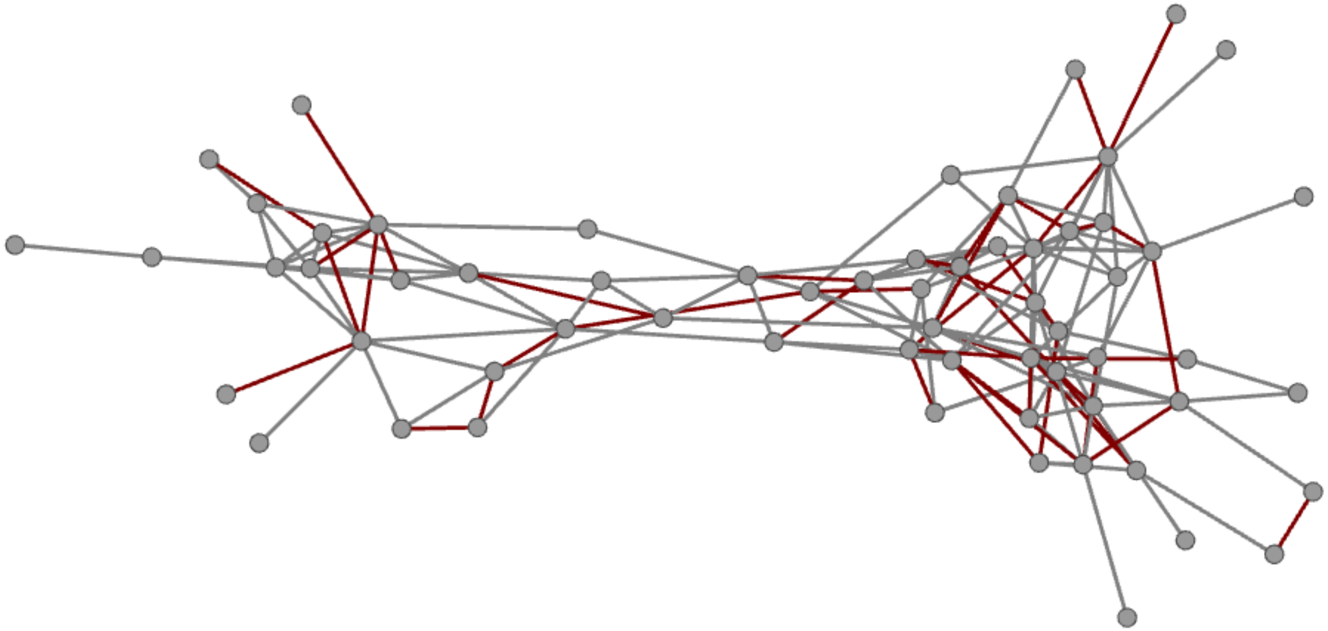
62 vertices, 159 edges

Search for a community of size 10 by maximizing

$$f(X) = |X| - \mathrm{rank}\,(X) \quad \textbf{if } |X| = \binom{10}{2} = 45$$

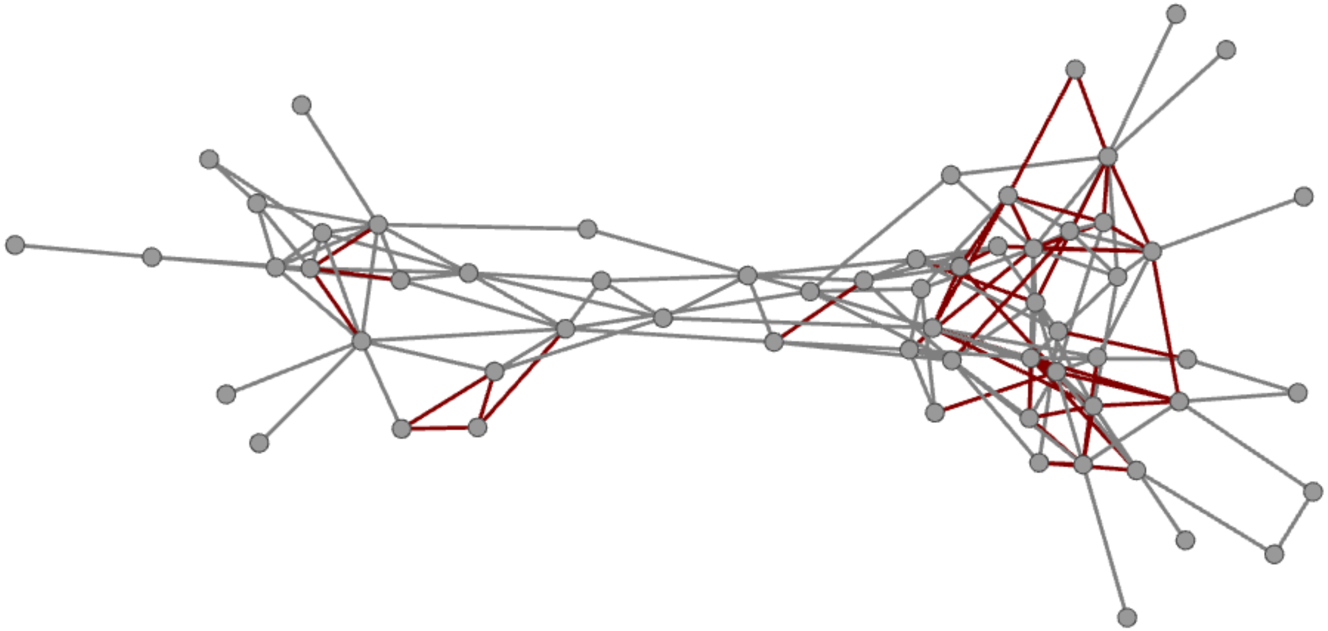


Step 0 (random): $f(X) = 3$

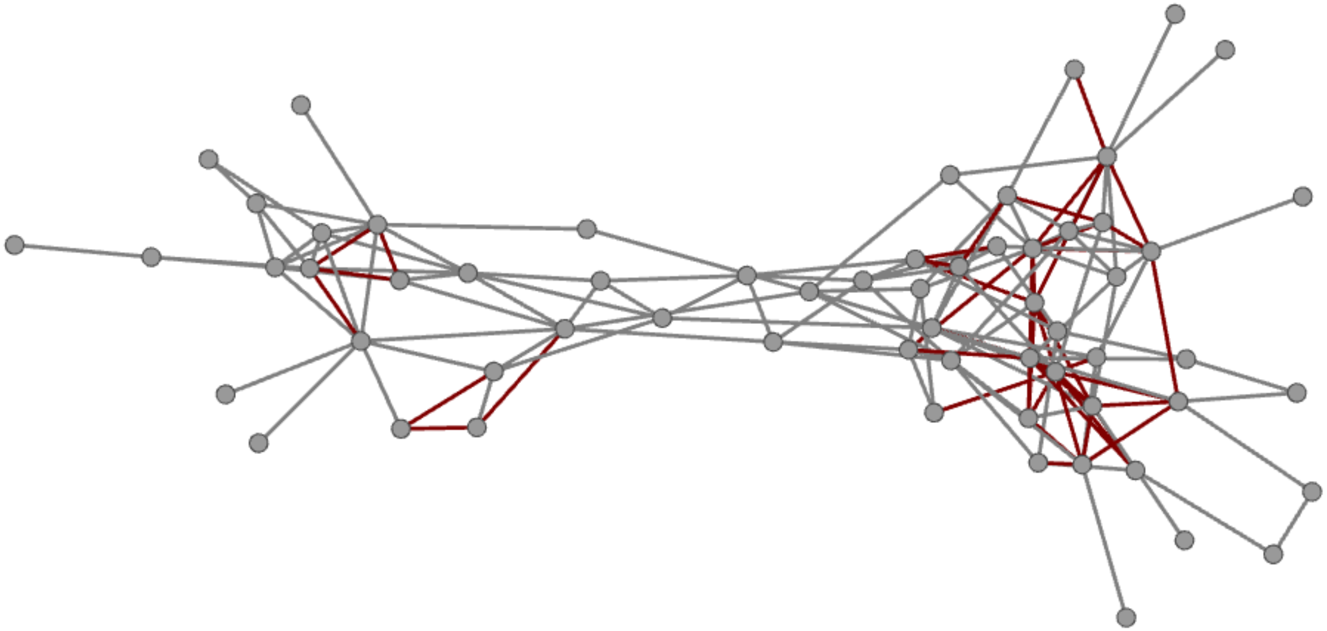


Visualized by Gephi (`https://gephi.github.io/`)

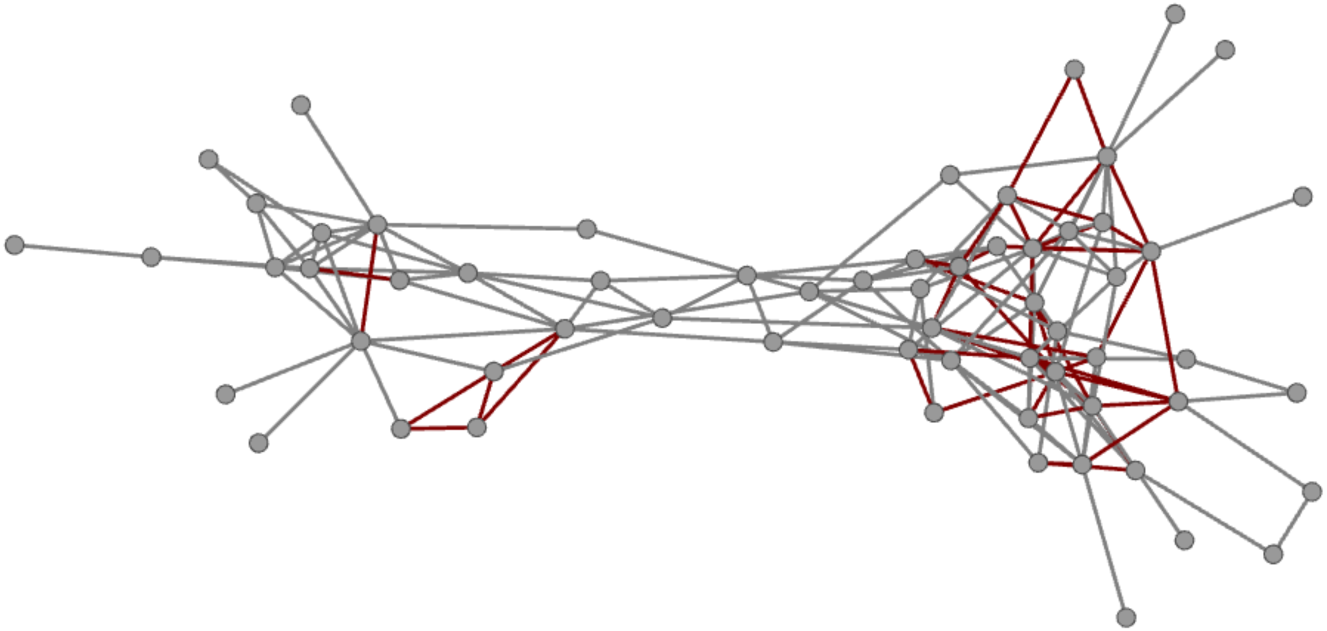
Step 1: $f(X) = 15$



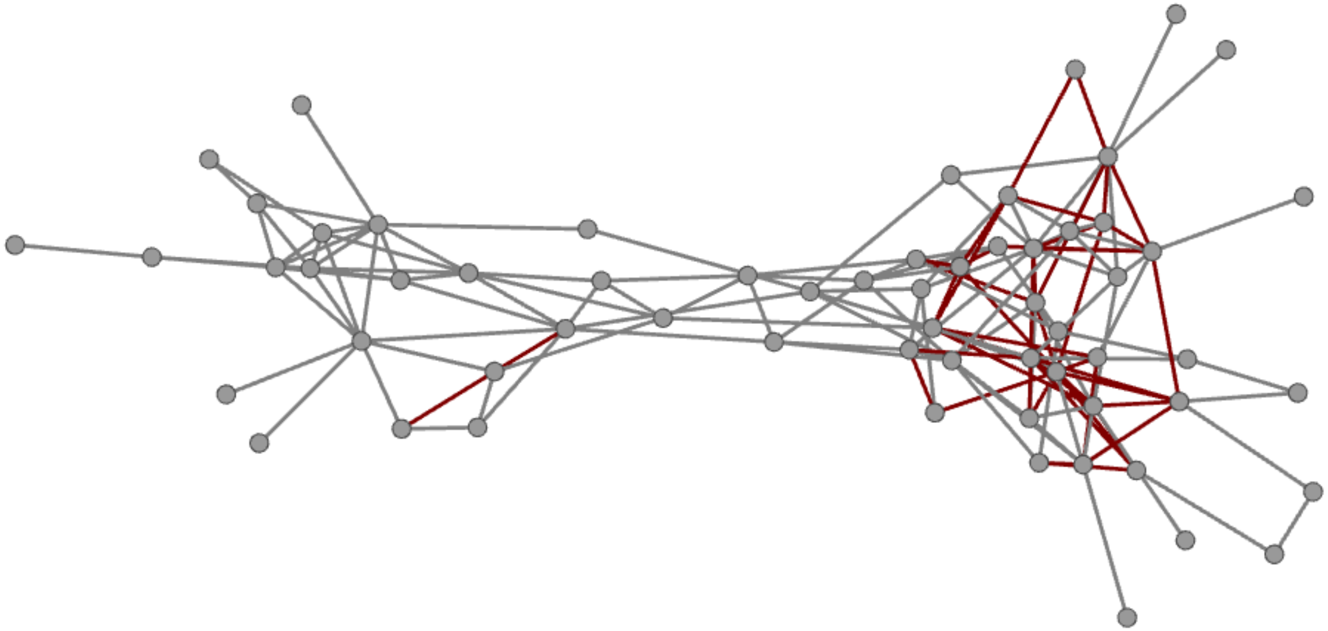
Step 2: $f(X) = 18$



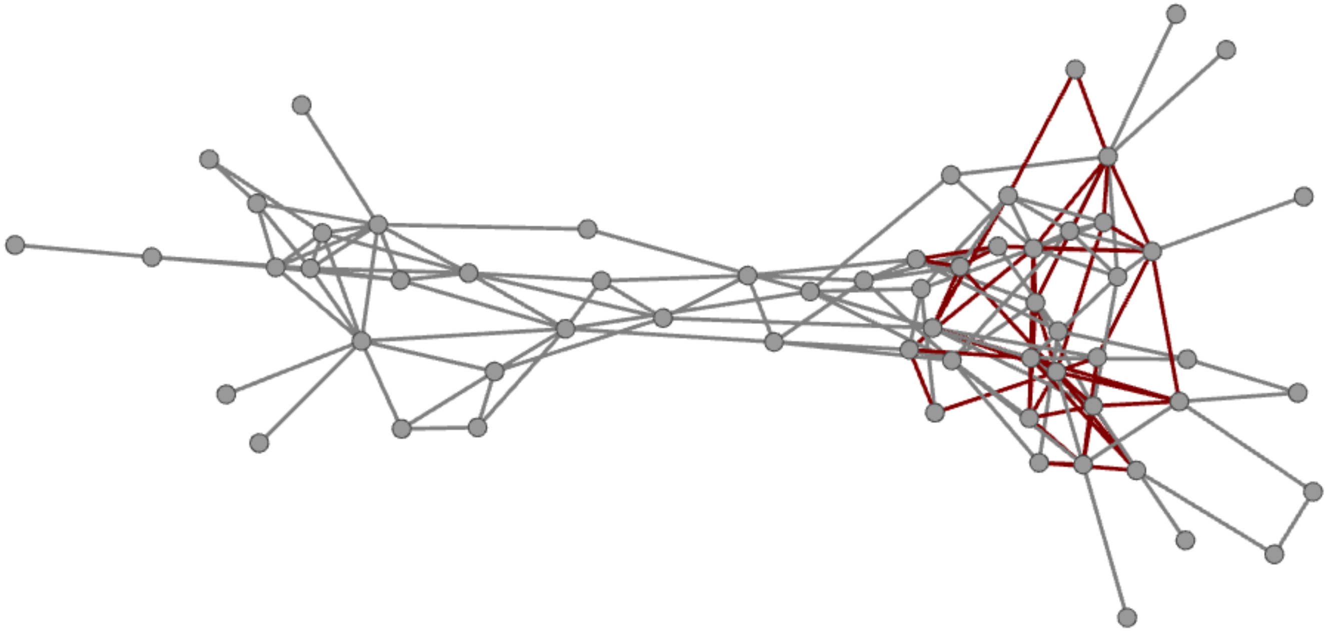
Step 4: $f(X) = 19$



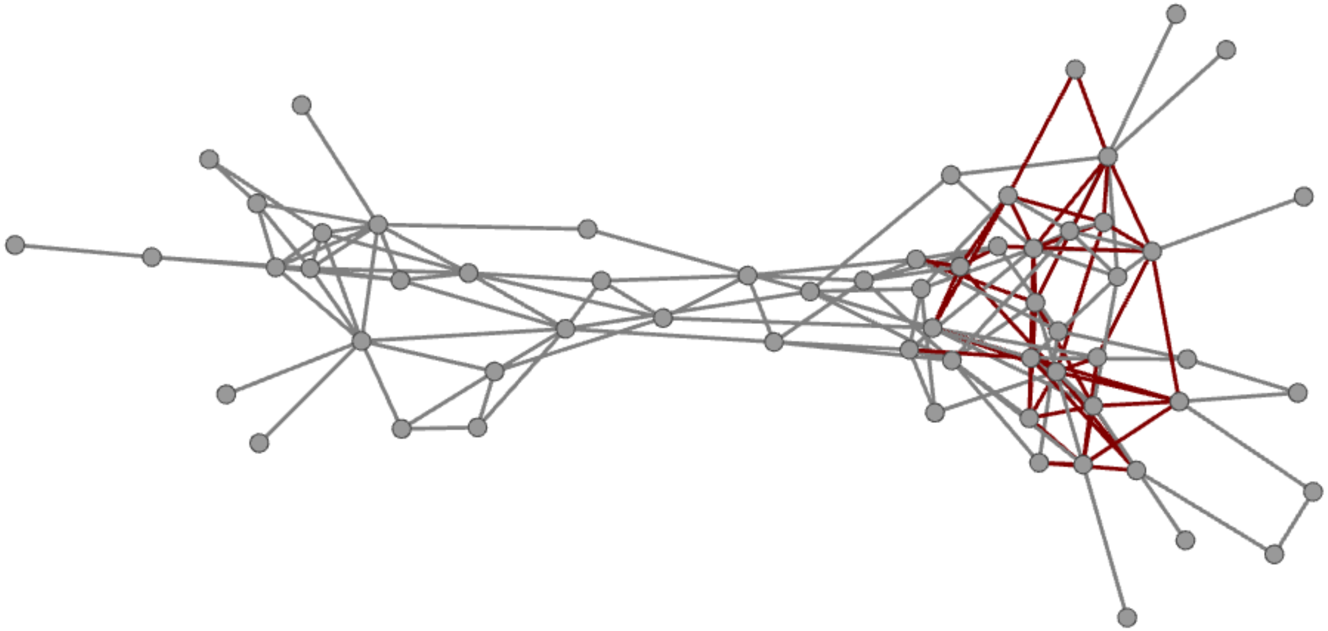
Step 5: $f(X) = 22$



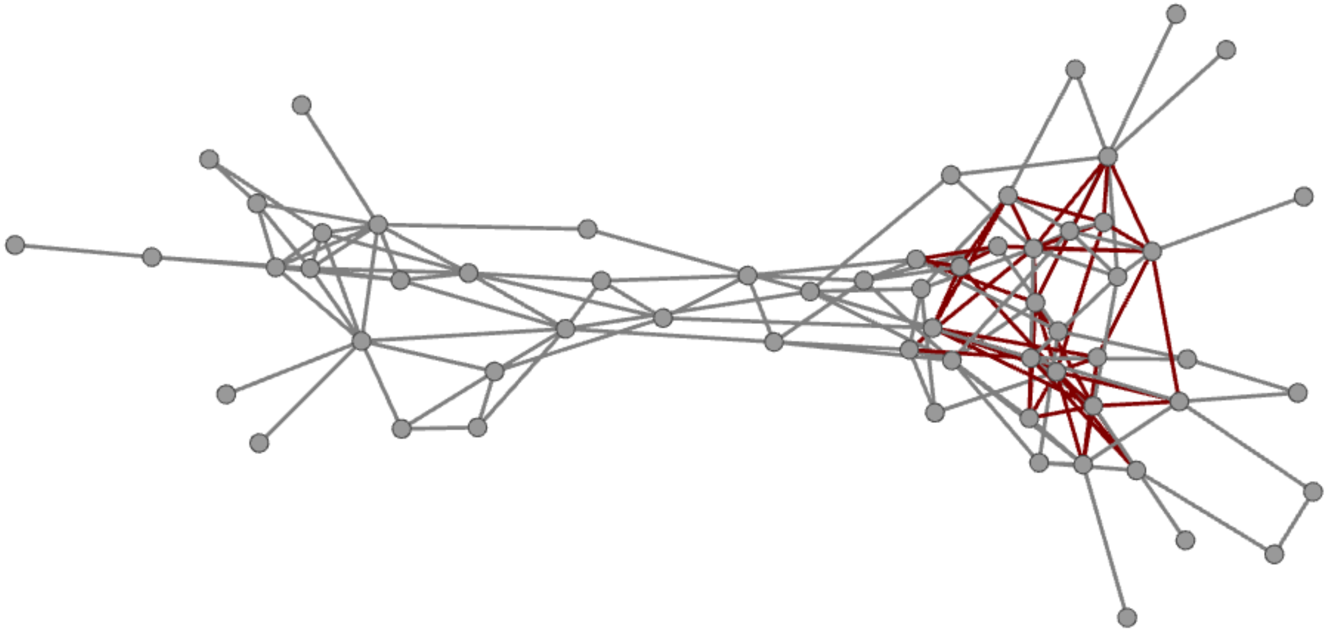
Step 6: $f(X) = 24$



Step 19: $f(X) = 25$



Step 21: $f(X) = 27$



Spanning Tree with Few Colors (M$^\natural$–M$^\natural$ DC)

$G = (V, E)$ graph, $\qquad E = \bigcup_{c \in C} E_c$ (color classes)

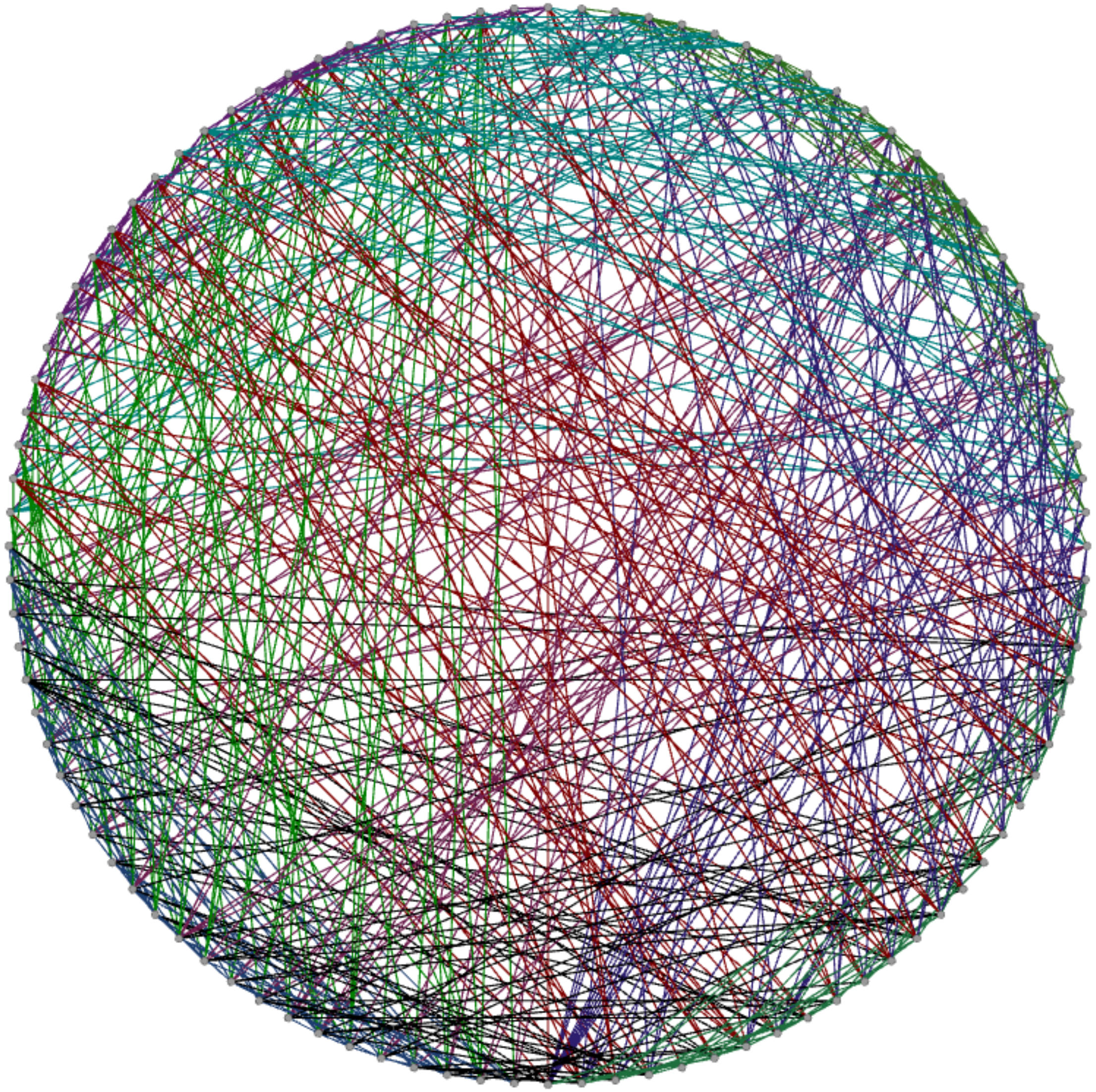
$$\text{Maximize } f(X) = \begin{cases} \sum_{c \in C} |E_c \cap X|^2, & X\text{: spanning tree} \\ -\infty, & \text{otherwise} \end{cases}$$

to obtain a spanning tree with few colors

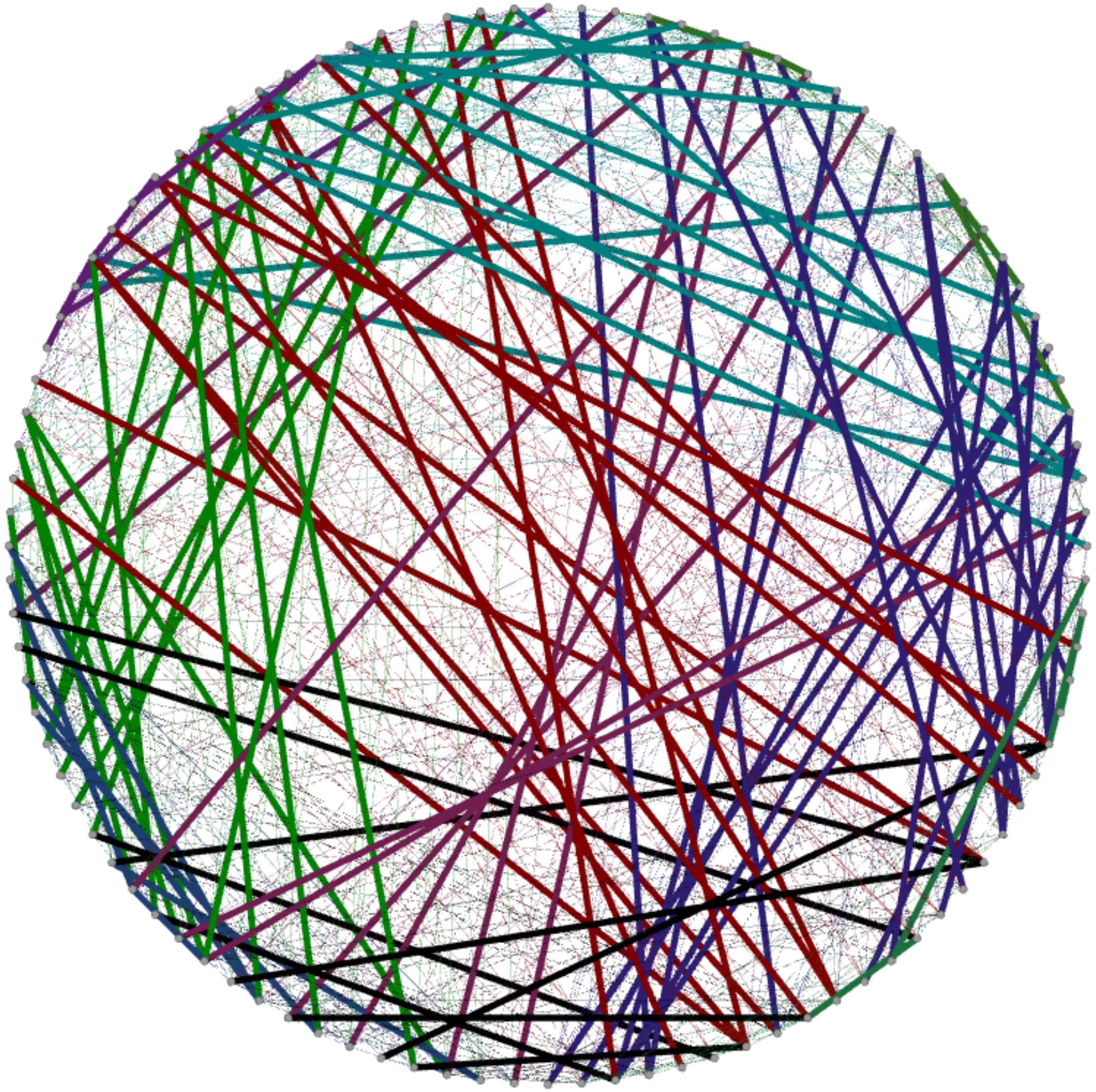
Erdős–Rényi graph $n = 100$, $p = 0.12$,

744 edges, 10 colors ($|C| = 10$)

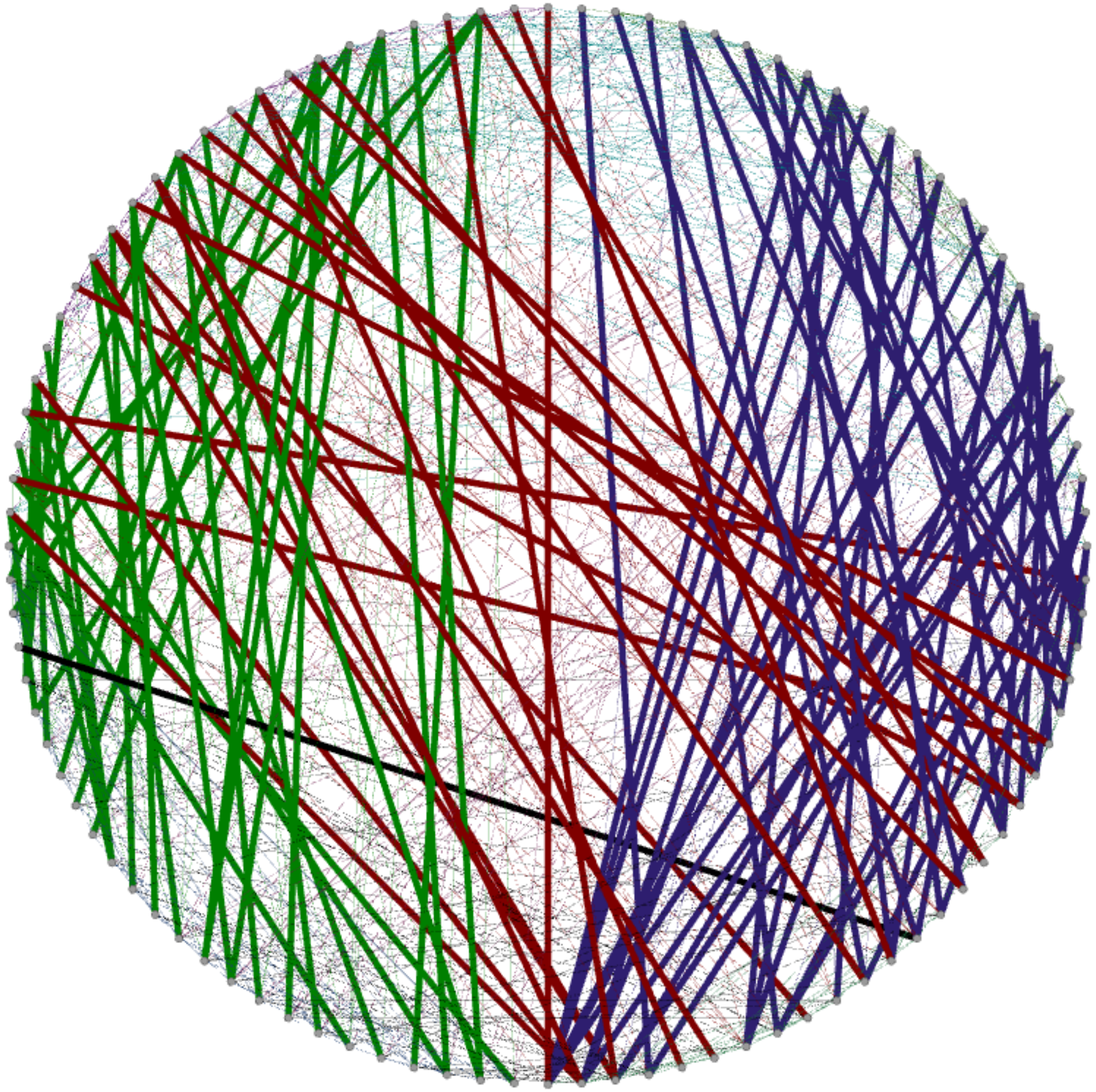
Input



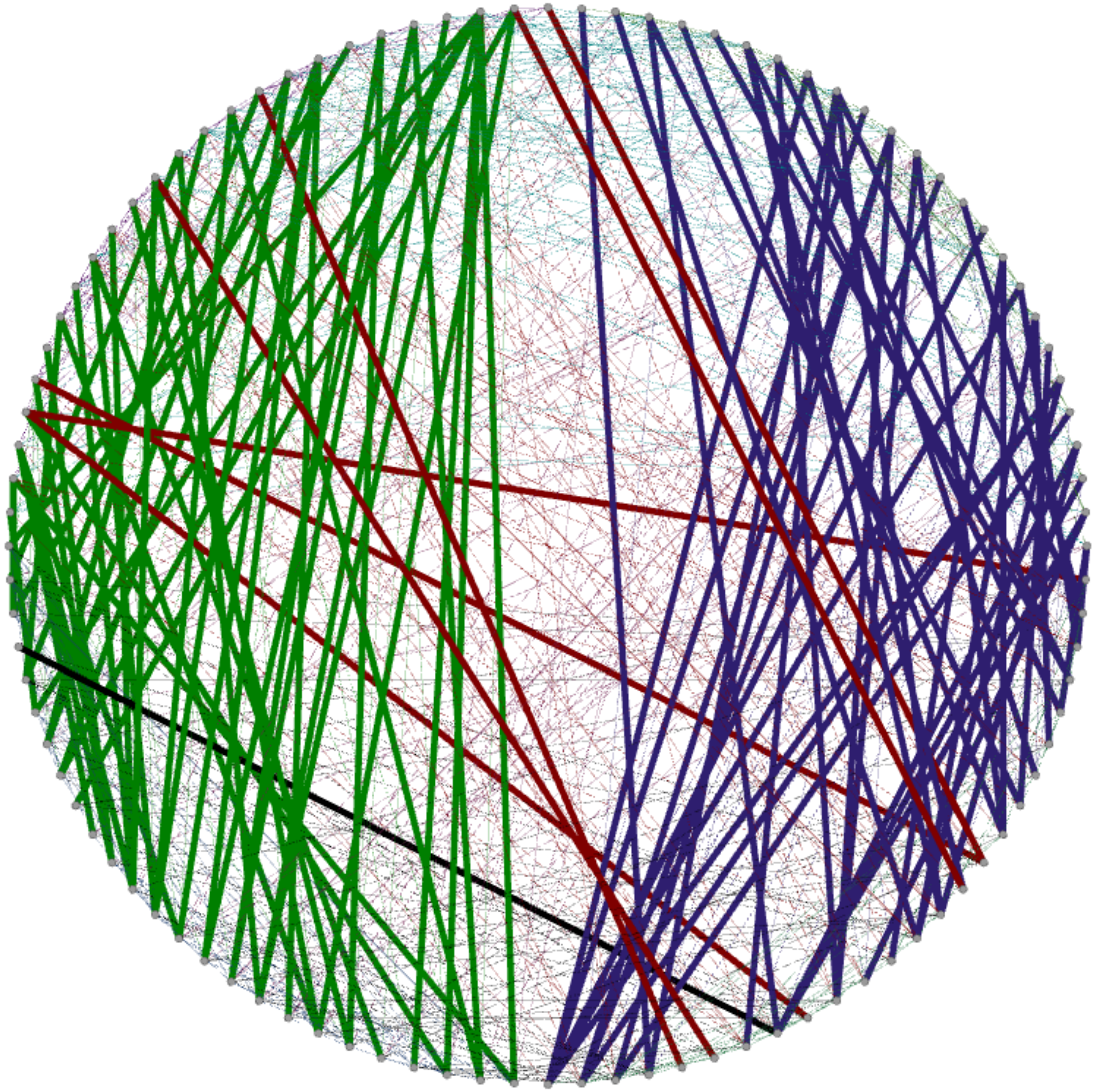
Step 0 (random) $f = 1221 \quad (19, 15, 12, 12, 9, 9, 9, 6, 4, 4)$



Step 1 $f = 3445$ $(44, 32, 22, 1, 0, 0, 0, 0, 0, 0)$



Step 2 $f = 4195 \quad (47, 44, 7, 1, 0, 0, 0, 0, 0, 0)$



Summary

- Framework of "discrete DC programming"
  M$^\natural$–M$^\natural$, L$^\natural$–M$^\natural$ (subm), M$^\natural$–L$^\natural$ (superm), L$^\natural$–L$^\natural$,
  integral subgradient, biconjugacy, Toland-Singer
- Local optimality condition
  $\partial g(x) \supseteq \partial h(x)$, checking this condition
- Algorithm
  monotone, finite, local opt
- Discrete convex analysis for nonconvex problems

Further study with

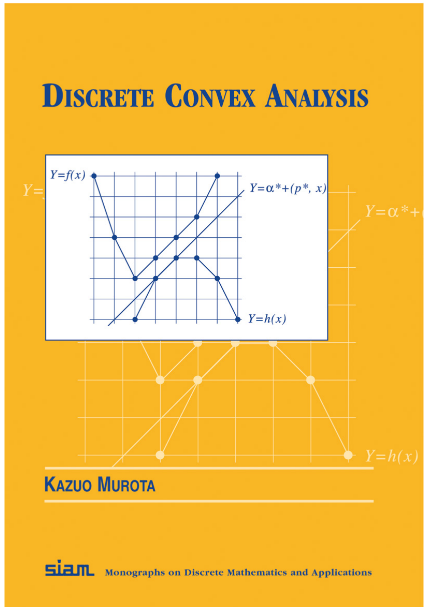
Takanori Maehara, Naoki Marumo, and ......

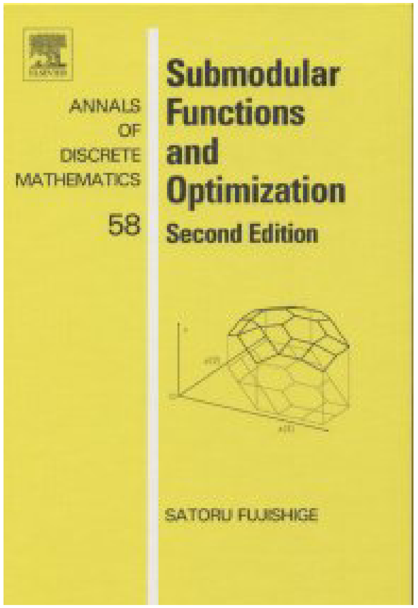
References

Maehara–Murota: A framework of discrete DC programming by discrete convex analysis, Math. Programming, 2014, on-line

Murota: Discrete Convex Analysis, SIAM, 2003

Fujishige: Submodular Functions and Optimization, 2nd ed., Elsevier, 2005 (Chap. VII)





Survey/Slide/Video/Software

[Survey]
Murota: Recent developments in discrete convex analysis (Research Trends in Combinatorial Optimization, Bonn 2008, Springer, 2009, 219–260)

[Slide] [Video]

http://www.misojiro.t.u-tokyo.ac.jp/ murota/publist.html#DCA

[Video]

https://smartech.gatech.edu/xmlui/handle/1853/43257/

https://smartech.gatech.edu/xmlui/handle/1853/43258/

[Software] DCP (Discrete Convex Paradigm)
http://www.misojiro.t.u-tokyo.ac.jp/DCP/

E N D