Hausdorff Institute of Mathematics, Summer School (September 21–25, 2015)
Problems for "Discrete Convex Analysis" (by Kazuo Murota)

Problem 1. Prove that a function $f : \mathbf{Z}^2 \to \mathbf{R}$ defined by $f(x_1, x_2) = \varphi(x_1 - x_2)$ is an $\mathrm{L}^\natural$-convex function, where $\varphi : \mathbf{Z} \to \mathbf{R}$ is a univariate discrete convex function (i.e., $\varphi(t-1) + \varphi(t+1) \geq 2\varphi(t)$ for all $t \in \mathbf{Z}$).

Problem 2. Prove that a function $f : \mathbf{Z}^2 \to \mathbf{R}$ defined by $f(x_1, x_2) = \varphi(x_1 + x_2)$ is an $\mathrm{M}^\natural$-convex function, where $\varphi : \mathbf{Z} \to \mathbf{R}$ is a univariate discrete convex function.

Problem 3. (1) Show that a function $f(x_1, x_2)$ is $\mathrm{M}^\natural$-convex if and only if $f(x_1, -x_2)$ is $\mathrm{L}^\natural$-convex.
(2) Is there any such correspondence for functions in three or more variables?

Problem 4. Prove that $f(x) = \max\{0, x_1, \ x_2, \ldots, \ x_n\}$ is an $\mathrm{L}^\natural$-convex function.

For a family $\mathcal{F}$ of subsets of $\{1, 2, \ldots, n\}$ and a family of univariate discrete convex functions $\varphi_A : \mathbf{Z} \to \mathbf{R}$ indexed by $A \in \mathcal{F}$, we consider a function defined by

$$f(x) = \sum_{A \in \mathcal{F}} \varphi_A(x(A)) \qquad (x \in \mathbf{Z}^n), \tag{1}$$

where $x(A) = \sum_{i \in A} x_i$. A function $f : \mathbf{Z}^n \to \mathbf{R}$ is called laminar convex if it can be represented in this form for some laminar family $\mathcal{F}$ and φ_A $(A \in \mathcal{F})$.

Problem 5. Prove that a laminar convex function is $\mathrm{M}^\natural$-convex.

In Problems 6–9, we consider a quadratic function in three variables $f(x) = x^\top A x$ $(x \in \mathbf{Z}^3)$ defined by a 3×3 symmetric matrix $A = (a_{ij})$.

Problem 6. (1) Find a necessary and sufficient condition on (a_{ij}) for $f(x)$ to be submodular.
(2) When $f(x)$ is submodular, is the matrix A positive semidefinite?

Problem 7. (1) Find a necessary and sufficient condition on (a_{ij}) for $f(x)$ to be $\mathrm{L}^\natural$-convex.
(2) When $f(x)$ is $\mathrm{L}^\natural$-convex, is the matrix A positive semidefinite?

Problem 8. (1) Show that $f(x)$ is an $\mathrm{M}^\natural$-convex function if and only if (i) $a_{ii} \geq a_{ij} \geq 0$ for all (i, j), and (ii) the minimum among the three off-diagonal elements, a_{12}, a_{23}, a_{13}, is attained by at least two elements.
(2) When $f(x)$ is $\mathrm{M}^\natural$-convex, is the matrix A positive semidefinite?

Problem 9. (1) Is $f(x_1, x_2, x_3) = (x_1 + x_2)^2 + (x_2 + x_3)^2 + (x_1 + x_3)^2$ laminar convex?
(2) Is this function $\mathrm{M}^\natural$-convex?
(3) Prove that a quadratic function $f(x)$ $(x \in \mathbf{Z}^3)$ is $\mathrm{M}^\natural$-convex if and only if it is laminar convex[1].

Problem 10. (1) Show that $f(x_1, x_2, x_3) = a(x_1 + x_2)^2 + b(x_2 + x_3)^2 + c(x_1 + x_3)^2$ with randomly chosen $a, b, c > 0$ is not an $\mathrm{M}^\natural$-convex function.
(2) Show that, under some "nondegeneracy assumption," a function $f(x)$ of the form (1) is $\mathrm{M}^\natural$-convex only if $\mathcal{F}$ is a laminar family.

[1]This statement is true for general n. That is, a quadratic function in n integer variables is $\mathrm{M}^\natural$-convex if and only if it is laminar convex.

Problem 11. A classical paper of Miller (1971) in inventory theory dealt with the function:

$$f(x) = \sum_{k=0}^{\infty} \left(1 - \prod_{i=1}^{n} F_i(x_i + k) \right) + \sum_{i=1}^{n} c_i x_i \qquad (x = (x_1, \dots, x_n) \in \mathbf{Z}_+^n), \tag{2}$$

where $F_1, \dots, F_n$ are cumulative distribution functions of Poisson distributions (with different means), and $c_1, \dots, c_n$ are nonnegative real numbers. Prove that this function is L$^\natural$-convex.

> The steepest descent algorithm for an L$^\natural$-convex function $g : \mathbf{Z}^n \to \mathbf{R} \cup \{+\infty\}$ reads as follows (e_X means the characteristic vector of a set $X \subseteq \{1, 2, \dots, n\}$):
> Step 0: Set $p := p^\circ$ (initial point).
> Step 1: Find $\sigma \in \{+1, -1\}$ and X that minimize $g(p + \sigma e_X)$.
> Step 2: If $g(p + \sigma e_X) = g(p)$, then output p and stop.
> Step 3: Set $p := p + \sigma e_X$ and go to Step 1.

In Problems 12 and 13 we consider the behavior of this algorithm when $n = 2$.

Problem 12. Define $g : \mathbf{Z}^2 \to \mathbf{R}$ by $g(p_1, p_2) = \max(0, -p_1 + 2, -p_2 + 1, -p_1 + p_2 - 1, p_1 - p_2 - 2)$.
(1) Verify that g is L$^\natural$-convex.
(2) Find the set, say, S of the minimizers of g. Draw a figure, indicating S on the lattice $\mathbf{Z}^2$.
(3) Take an initial point $p^\circ = (0, 0)$. Which minimizers are possibly found? Is the number of iterations constant, independent of the generated sequences of vector p? How is the number of iterations related to the ℓ_∞-distance from p° to S?
(4) Take another initial point $p^\circ = (1, 4)$. Which minimizers are possibly found? Is the number of iterations equal to the ℓ_∞-distance from p° to S?

Problem 13. Let $g : \mathbf{Z}^2 \to \mathbf{R}$ be an L$^\natural$-convex function that has a minimizer; denote by S the set of its minimizers. Give an expression for the number of iterations in terms of p° and S.

Problem 14 (M-minimizer cut theorem)**.** Let $f : \mathbf{Z}^n \to \mathbf{R}$ be an M-convex function such that $\mathrm{argmin}\, f \neq \emptyset$. Take any $x \in \mathrm{dom}\, f$ and $i \in \{1, 2, \dots, n\}$, and let $j \in \{1, 2, \dots, n\}$ be such that $f(x - e_i + e_j) = \min_{1 \le k \le n} f(x - e_i + e_k)$. Prove that there exists $x^* \in \mathrm{argmin}\, f$ such that $x_j^* \ge x_j + 1$ in the case of $i \neq j$ and $x_j^* \ge x_j$ in the case of $i = j$.

For a matroid on V, the rank function ρ is defined by

$$\rho(X) = \max\{|I| \mid I \text{ is an independent set, } I \subseteq X\} \qquad (X \subseteq V). \tag{3}$$

Problem 15. Let ρ be a matroid rank function on V, and identify ρ with a function $f : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ defined by $f(e_X) = \rho(X)$ for $X \subseteq V$ with $\mathrm{dom}\, f = \{0, 1\}^V$.
(1) Prove that ρ is L$^\natural$-convex.
(2) Prove that ρ is M$^\natural$-concave.
(3) Prove that $f(e_X) + f^\bullet(e_X) = |X|$ for $X \subseteq V$, where $f^\bullet : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ is the (convex) discrete Legendre transform of f.

Problem 16. Let ρ_1 and ρ_2 be the rank functions of two matroids on V. For the rank of the union matroid, the following formula is known:

$$\max_X \{\rho_1(X) + \rho_2(V \setminus X)\} = \min_Y \{\rho_1(Y) + \rho_2(Y) - |Y|\} + |V|. \tag{4}$$

Relate this formula to the Fenchel min-max duality in discrete convex analysis.

Problem 100 (Research Problem)**.** Let $G = (V, W; E)$ be a bipartite graph with edge cost $c : E \to \mathbf{R}$. Suppose that a matroid is given on V, with $\mathcal{I}$ denoting the family of independent sets. For $Y \subseteq W$ define $f(Y)$ as the minimum cost of a matching that respects the matroid on V and matches with Y on W:

$$f(Y) = \min\{\sum_{e \in M} c(e) \mid M \text{ is a matching}, V \cap \partial M \in \mathcal{I}, W \cap \partial M = Y\}, \tag{5}$$

where $f(Y) = +\infty$ if no such M exists. It is known that this f is an M$^\natural$-convex set function. Does every M$^\natural$-convex set function f with $f(\emptyset) = 0$ arise in this way? That is, given an M$^\natural$-convex set function f on W with $f(\emptyset) = 0$, can we find a bipartite graph $G = (V, W; E)$, a cost function $c : E \to \mathbf{R}$, and a matroid on V for which the above construction yields the given function f?