Quantitative Analysis for Controllability of Symmetric Control Systems

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This paper is concerned with the controllability of systems possessing symmetries. In particular, we aim to give a lower bound for the number of functioning modules needed to ensure the controllability of the entire symmetric system. Our concern is the characteristics derived from the symmetry of the systems, and not from the numerical information nor from the sparsity of the connections among the modules. If structurally independent modules are connected, only one functioning module would suffice to ensure the controllability. For symmetric systems, however, it is shown that we need more than one functioning module for the entire system to be controllable. We treat systems of general symmetry and show the lower bound based on group representation theory.

1. Introduction

In the field of control theory, scattered researches have been carried out concerning the control of group-theoretic symmetric systems (1; 2; 3; 4; 5; 6; 10; 11). Symmetry in systems arises as a result of natural design, and provides a pivotal mechanism for crucial system properties. Symmetric systems are amenable to theoretical analysis because of considerable simplifications in modelling and analysis. One of the major theoretical tools for the analysis of symmetric systems is the method of group-theoretic decomposition into smaller subsystems as shown in (3). Examples of such theoretical treatment of symmetric systems include (8; 13) which deal with systems consisting of symmetrically connected identical subsystems, and (3) which deal with the twin lift (two helicopters) concept to lift a mass. Based on the same decomposition method, this paper focuses on the controllability of symmetric systems. In particular, we will be concerned with systems consisting of a number of modules, each of which has its own control input, such as decentralized control systems.

For decentralized control systems in general, the controllability should be retained even with limited number of control inputs (12; 16). The limitation on the number of control inputs is introduced for the cost efficiency or for the ease of designing controllers. Then, it is natural to investigate the lower bound for the number of control inputs, i.e., that of functioning modules, needed to keep the entire system controllable. Such investigation serves as a quantitative analysis for the controllability problem of decentralized systems.

For analysis of large-scale systems, it is useful to focus on their structure (in a general sense), rather than on the numerical information. Symmetry of systems is undoubtedly one of such structures to be considered. This paper, in particular, deals with symmetric systems with respect to controllability. That is, we aim to give a lower bound for the number of functioning modules needed to keep the symmetric system controllable. It should be mentioned, in this connection, that the fundamental result on structural controllability (7) would imply a trivial solution that only one functioning module suffices to keep the controllability if structurally independent modules are connected. However, symmetry consideration as a system structure brings us more meaningful

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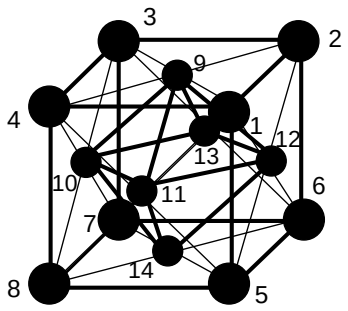

Figure 1.. A cube with a regular octahedron inside.

figure

results. For example, the cubic system shown in Fig. 1, geometrically symmetric as is seen from the figure, requires at least five functioning modules for the controllability as a result of the symmetry. The cubic systems with less than five control inputs can never be controllable. It is emphasized that the lower bound is obtained from the group-theoretic symmetry.

Our analysis of the lower bound is based on the standard group-theoretic decomposition technique of symmetric systems, in corporation with the introduction of the functioning pattern of the modules and the orbit decomposition of a system. It should be noted that the modules in different orbits are not needed to be identical, and therefore, this paper deals with larger class of systems than those consisting of identical subsystems (8; 13).

The outline of this paper is as follows. In Section 2, the formulation of the systems treated in this paper is described. Section 3 shows the main result that gives a lower bound for the number of the functioning modules needed to ensure the controllability of the entire symmetric system. In Section 4, the main result is illustrated for systems with different symmetries. Finally, concluding remarks are given in Section 5. Preliminaries from group representation theory are given in Appendix.

2. Formulation of Symmetric Systems

Consider a linear time-invariant system $\mathcal{S}$ that consists of m control modules $\{\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_m\}$, each of which has its own control channel. The entire system $\mathcal{S}$ is then described by

$$\begin{bmatrix} \dot{\boldsymbol{x}}_1 \\ \dot{\boldsymbol{x}}_2 \\ \vdots \\ \dot{\boldsymbol{x}}_m \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & \cdots & A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mm} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \vdots \\ \boldsymbol{x}_m \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1m} \\ B_{21} & B_{22} & \cdots & B_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ B_{m1} & B_{m2} & \cdots & B_{mm} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \\ \vdots \\ \boldsymbol{u}_m \end{bmatrix}, \tag{2.1}$$

where $\boldsymbol{x}_i(t) \in \mathbf{R}^{n_i}$ and $\boldsymbol{u}_i(t) \in \mathbf{R}^{r_i}$ denote the state of $\mathcal{S}_i$ and the input from its control channel, respectively, with $\mathbf{R}^n$ being the set of real vectors of the size n. The diagonal elements A_{ii} and B_{ii} in (2.1) represent the effects of each module $\mathcal{S}_i$ on its own state $\boldsymbol{x}_i$. The off-diagonal elements A_{ij} and B_{ij} $(i \neq j)$ imply the interactions among modules through the interconnections. Therefore, if two modules $\mathcal{S}_i$ and $\mathcal{S}_j$ $(i \neq j)$ are not connected, the corresponding matrices A_{ij} and B_{ij} are zero. By denoting the state and the input of the entire system $\mathcal{S}$ as

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \vdots \\ \boldsymbol{x}_m \end{bmatrix} \in \mathbf{R}^n, \quad \boldsymbol{u} = \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \\ \vdots \\ \boldsymbol{u}_m \end{bmatrix} \in \mathbf{R}^r, \tag{2.2}$$

respectively, the equation (2.1) can be given in the standard form of a state transition equation:

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t). \tag{2.3}$$

Among the systems described by (2.3), we are interested in ones with group-theoretic symmetry. We say that the system (2.3) is symmetric with respect to a finite group G if

$$T(g)A = AT(g), \quad T(g)B = BS(g), \quad g \in G, \tag{2.4}$$

where T and S are unitary representations of G on $\mathbf{R}^n$ and $\mathbf{R}^r$, respectively (see e.g., (9) Appendix for terminology from group representation theory). The equation (2.4) often reflects the underlying geometric symmetry in the system structures.

We will be concerned with the minimum number of functioning modules needed to ensure the controllability of the entire symmetric system. Let M and N denote the index sets of the functioning modules and of the rest of the modules, respectively. The functioning pattern of the system is thus described by the pair of M and N. In addition, we introduce the functioning pattern matrix F of order r in such a way that the matrix BF has zero column blocks that correspond to the control channels which are not functioning. Such a matrix F is given by $F = \bigoplus_{i=1}^{m} F_i$ with

$$F_i = \begin{cases} I_{r_i} & (i \in M), \\ O_{r_i} & (i \in N), \end{cases}$$

where the matrices I_k and O_k denote, respectively, the unit matrix and the zero matrix of order k in general. This means

$$F = F_M \oplus F_N = I_f \oplus O_{r-f} \tag{2.5}$$

with $F_M = \bigoplus_{i\in M} F_i = I_f$, $F_N = \bigoplus_{i\in N} F_i = O_{r-f}$, $f = \sum_{i\in M} r_i$, by an appropriate permutation of the indices of the modules. Note that any functioning pattern can be given in the form of (2.5) and that the system to be considered is denoted as (A, BF).

It should be noted here that we are interested in the characteristics of a system determined by its symmetric structure and not by the numerical information of the system matrices A and B. More precisely, we are interested in a generic system subject to the symmetry constraint defined in terms of the equation (2.4) (see Remark 2 in Section 3 for precise definition of the genericity under the symmetry condition). However, our main result, shown in the next section, does not rely on the genericity assumption.

3. Main Result

This section clarifies the minimum number of functioning modules needed to ensure the controllability of the entire G-symmetric system. The main theorem of this paper (Theorem 1) is given in subsection 3.1, whereas its derivation is shown in subsection 3.2.

3.1 *The Covering Condition*

We shall give a theorem which gives a lower bound for the number of functioning modules needed to keep the controllability of the entire G-symmetric system. The lower bound is verified to be often tight by examples.

In the following, we consider the state space $\mathcal{X} \simeq \mathbf{C}^n$ and the input space $\mathcal{U} \simeq \mathbf{C}^r$ for the simplicity of mathematical treatment[3], although the systems formulated above are described on real spaces. The family of all nonequivalent irreducible matrix representations of G is denoted by

[3]Namely, we consider the complexifcations of $\mathcal{X}$ and $\mathcal{U}$.

$\{D^\mu \mid \mu \in R(G)\}$, where D^μ is an unitary irreducible matrix representation, of N^μ-dimension, on $\mathbf{C}$ and $R(G)$ is the index set for the complex irreducible representations of G. Moreover, we assume that the representations T and S in (2.4) are permutation representations that stand for permutations of the modules, and are decomposed as direct sums of irreducible representations as

$$T = \sum_{\mu \in R(G)} a^\mu \mu, \quad S = \sum_{\mu \in R(G)} b^\mu \mu, \tag{3.1}$$

where a^μ and b^μ are multiplicities.

The discussion will be based on the orbit decomposition of the system. For simplicity of discussion, we deal with the case where $n_i = 1$ and $r_i = 1$ $(1 \leq i \leq m)$, i.e., each module has a one-dimensional state and input. However, a similar argument is possible for systems with higher-dimensional modules.

We denote by P the index set of the modules in the original system. Let

$$P = \bigcup_{l=1}^{L} P_l \tag{3.2}$$

be the decomposition of P into L disjoint orbits with respect to the action of G, the permutations of the indices of the modules. The permutation representations T and S in (2.4) are decomposed accordingly as

$$T(g) = \bigoplus_{l=1}^{L} T_l(g), \quad S(g) = \bigoplus_{l=1}^{L} S_l(g), \quad g \in G. \tag{3.3}$$

The matrices $T_l(g)$ and $S_l(g)$ $(1 \leq l \leq L)$ are decomposed further as direct sums of irreducible representations as

$$T_l = \sum_{\mu \in R(G)} a_l^\mu \mu, \quad S_l = \sum_{\mu \in R(G)} b_l^\mu \mu, \tag{3.4}$$

where a_l^μ and b_l^μ are multiplicities, and satisfy $\sum_{l=1}^{L} a_l^\mu = a^\mu$, $\sum_{l=1}^{L} b_l^\mu = b^\mu$.

Recall that we denote by M the index set of the functioning modules and $|M| = f$. Therefore, similarly to (3.2),

$$M = \bigcup_{l=1}^{L} M_l, \tag{3.5}$$

where $M_l = M \cap P_l$ $(1 \leq l \leq L)$, gives the decomposition of M into L disjoint sets. Hence

$$|M| = \sum_{l=1}^{L} |M_l| \tag{3.6}$$

is satisfied.

With these orbit decompositions, the following theorem provides a lower bound for the number of functioning modules $|M|$ needed to keep the entire system controllable. Its proof will be shown in the next subsection.

Theorem 1. If a G-symmetric system (A, B) is controllable with the functioning modules indexed by $M = \bigcup_{l=1}^{L} M_l$, then

$$\sum_{l, a_l^\mu \neq 0} |M_l| \geq N^\mu \tag{3.7}$$

holds for all $\mu \in R(G)$ with $a^\mu \neq 0$. □

We call the condition (3.7) the *covering condition*. It should be emphasized that the covering condition is obtained systematically from the symmetric group of the system to be considered. A

lower bound for $|M|$ is then obtained from the covering condition with the restrictions that $|M_l|$ are integers, namely,

$$|M| \geq \min\{\sum_{l=1}^{L} |M_l| \mid \{M_l\}_{l=1}^{L} \text{ satisfies}(3.7), |M_l| : \text{integers}\}. \tag{3.8}$$

Theorem 1 yields a simpler lower bound for $|M|$, as follows.

Corollary 1. If a G-symmetric system (A, B) is controllable with the functioning modules indexed by M, then

$$|M| \geq \max_{\mu \in R(G), a^\mu \neq 0} N^\mu. \tag{3.9}$$

□

(Proof) If the system (A, BF) is controllable, the covering condition (3.7) holds for all $\mu \in R(G)$ with $a^\mu \neq 0$ by Theorem 1. Since $|M| \geq \sum_{l, a_l^\mu \neq 0} |M_l|$ holds by (3.6), the inequality (3.9) holds. ■

It should be noted that the right-hand side of (3.8) is always greater than or equal to that of (3.9). This means that the lower bound obtained from the covering condition is no worse than the lower bound (3.9). Furthermore, the former often provides us with a better (strictly larger) lower bound than the latter, as we will see in Section 4. The tightness of the lower bound (3.8) should be verified by specifying functioning modules, for example, with the help of Theorem 3 of (15).

3.2 *Proofs*

The covering condition (3.7) is derived in this subsection in three steps. Firstly, the standard decomposition method of a symmetric system into a direct sum of subsystems, as described in (3), is explained. Secondly, we incorporate the functioning pattern F into this decomposition to show Lemma 4, a necessary condition for a G-symmetric system to be controllable. Finally, Theorem 1 will be proved based on Lemma 4.

The spaces $\mathcal{X}$ and $\mathcal{U}$ are decomposed as direct sums of the invariant subspaces corresponding to D^μ as

$$\mathcal{X} = \bigoplus_{\mu \in R(G)} \bigoplus_{i=1}^{a^\mu} \mathcal{X}_i^\mu, \quad \mathcal{U} = \bigoplus_{\mu \in R(G)} \bigoplus_{j=1}^{b^\mu} \mathcal{U}_j^\mu. \tag{3.10}$$

We define two unitary matrices Z of order n and W of order r as

$$Z = (Z^\mu \mid \mu \in R(G)), \quad W = (W^\mu \mid \mu \in R(G)), \tag{3.11}$$

where

$$Z^\mu = (Z_i^\mu \mid 1 \leq i \leq a^\mu) \in \mathbf{C}^{n \times a^\mu N^\mu}, \quad W^\mu = (W_j^\mu \mid 1 \leq j \leq b^\mu) \in \mathbf{C}^{r \times b^\mu N^\mu}, \tag{3.12}$$

with $Z_i^\mu \in \mathbf{C}^{n \times N^\mu}$ and $W_j^\mu \in \mathbf{C}^{r \times N^\mu}$ being sets of bases of $\mathcal{X}_i^\mu$ and $\mathcal{U}_j^\mu$, respectively. Note that the unitarity of T and S allows us to choose the matrices Z and W to be unitary over $\mathbf{C}$, i.e., $Z^*Z = I_n$ and $W^*W = I_r$, where Z^* and W^* are the transposed conjugate matrices of Z and W, respectively. Since Z_i^μ and W_j^μ are bases of $\mathcal{X}_i^\mu$ and $\mathcal{U}_j^\mu$, respectively,

$$T(g)Z_i^\mu = Z_i^\mu D^\mu(g), \quad S(g)W_j^\mu = W_j^\mu D^\mu(g), \quad g \in G, \tag{3.13}$$

holds and thus

$$T(g)Z^\mu = Z^\mu(\bigoplus_{i=1}^{a^\mu} D^\mu(g)), \quad S(g)W^\mu = W^\mu(\bigoplus_{j=1}^{b^\mu} D^\mu(g)), \quad g \in G, \tag{3.14}$$

where $\bigoplus_{i=1}^{a^\mu} D^\mu(g)$ denotes a block-diagonal matrix consisting of a^μ identical diagonal blocks $D^\mu(g)$. Then the representations $T(g)$ and $S(g)$ of G are decomposed by Z and W into diagonal blocks of irreducible representations as

$$Z^*T(g)Z = \bigoplus_{\mu\in R(G)} \bigoplus_{i=1}^{a^\mu} D^\mu(g), \quad W^*S(g)W = \bigoplus_{\mu\in R(G)} \bigoplus_{j=1}^{b^\mu} D^\mu(g), \quad g \in G. \tag{3.15}$$

From (2.4) and (3.15), the system matrices A and B are also block diagonalized by the same matrices Z and W, from Schur's lemma (see Appendix for details), as

$$\tilde{A} = Z^*AZ = \bigoplus_{\mu\in R(G)} \bigoplus_{k=1}^{N^\mu} A^\mu, \quad \tilde{B} = Z^*BW = \bigoplus_{\mu\in R(G)} \bigoplus_{k=1}^{N^\mu} B^\mu, \tag{3.16}$$

with matrices $A^\mu \in \mathbf{C}^{a^\mu\times a^\mu}$ and $B^\mu \in \mathbf{C}^{a^\mu\times b^\mu}$. Note that $\bigoplus_{k=1}^{N^\mu} A^\mu$ in (3.16) denotes a block-diagonal matrix consisting of N^μ identical diagonal blocks A^μ. The block-diagonalization (3.16) describes the decomposition into a direct sum of subsystems.

As the second step to derive Theorem 1, we incorporate the functioning pattern F into the decomposition (3.16). The controllability of the G-symmetric system (A, B) with a functioning pattern F is equivalent to that of the system

$$(\tilde{A}, \tilde{B}W^*F) = \Big(\bigoplus_{\mu\in R(G)} \bigoplus_{k=1}^{N^\mu} A^\mu, \bigoplus_{\mu\in R(G)} ((\bigoplus_{k=1}^{N^\mu} B^\mu)(W^\mu)^*F)\Big), \tag{3.17}$$

where W and W^μ are defined in (3.11) and (3.12), respectively. The matrix $(W^\mu)^*F$ in (3.17) is denoted as

$$(W^\mu)^*F = \left[\, (W_M^\mu)^* \;\; O \,\right] \tag{3.18}$$

with the functioning matrix F in the form of (2.5), where W_M^μ is the first f rows of W^μ in

$$W^\mu = \begin{pmatrix} W_M^\mu \\ W_N^\mu \end{pmatrix}. \tag{3.19}$$

The discussion above, together with the well-known lemma below, leads us to Lemma 2.

Lemma 1. Suppose the matrices A and B for a system (A, B) are given as

$$A = \begin{bmatrix} A_1 & O \\ O & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},$$

with square submatrices A_1 and A_2. If the system (A, B) is controllable, then two subsystems (A_1, B_1) and (A_2, B_2) are controllable. The converse is also true if A_1 and A_2 have no common eigenvalues. □

Lemma 2. If a G-symmetric system (A, B) is controllable with a functioning pattern F, then $(\bigoplus_{k=1}^{N^\mu} A^\mu, (\bigoplus_{k=1}^{N^\mu} B^\mu)(W^\mu)^*F)$ is controllable for each μ satisfying $a^\mu \neq 0$. □

Accordingly, we are to investigate a necessary condition for the controllability of the subsystem $(\bigoplus_{k=1}^{N^\mu} A^\mu, (\bigoplus_{k=1}^{N^\mu} B^\mu)(W_M^\mu)^*)$ for μ with $a^\mu \neq 0$.

A necessary condition for the controllability of a system $(\bigoplus_{k=1}^{N} A, (\bigoplus_{k=1}^{N} B)W^*)$ in general is given in the following, which serves as a key technical lemma in the derivation of Theorem 1. The controllability of a system (A, B) is proved with reference to the rank of $\mathcal{C}_{\rm m}(A, B)$, where $\mathcal{C}_{\rm m}(A, B) = [A - \lambda I \mid B]$, $\lambda \in \mathbf{C}$, is the modal controllability matrix of the system (A, B).

Lemma 3. Consider a system $(\bar{A}, \bar{B})$ with the matrices $\bar{A}$ and $\bar{B}$ given as

$$\bar{A} = \bigoplus_{k=1}^{N} A, \quad \bar{B} = (\bigoplus_{k=1}^{N} B)W^*, \tag{3.20}$$

where $A \in \mathbf{C}^{a\times a}, B \in \mathbf{C}^{a\times b}$, and $W = (W^k \mid 1 \le k \le N) \in \mathbf{C}^{f\times bN}$ with $W^k = (\boldsymbol{w}_j^k \mid 1 \le j \le b) \in \mathbf{C}^{f\times b}$. If the system $(\bar{A}, \bar{B})$ is controllable, then there exists a set of N indices $\{j_1, j_2, \ldots, j_N\}$ $(1 \le j_k \le b, 1 \le k \le N)$ such that

$$\text{rank} \begin{bmatrix} (\boldsymbol{w}_{j_1}^1)^* \\ (\boldsymbol{w}_{j_2}^2)^* \\ \vdots \\ (\boldsymbol{w}_{j_N}^N)^* \end{bmatrix} = N. \tag{3.21}$$

□

(Proof) The system $(\bar{A}, \bar{B})$ is given as

$$(\bar{A}, \bar{B}) = (\begin{bmatrix} A & & & \\ & A & & \\ & & \ddots & \\ & & & A \end{bmatrix}, \begin{bmatrix} B(W^1)^* \\ B(W^2)^* \\ \vdots \\ B(W^N)^* \end{bmatrix}). \tag{3.22}$$

By Lemma 1, if the entire system is controllable, so is the subsystem $(\bigoplus_{k\in K} A, (\bigoplus_{k\in K} B)(W_K)^*)$ for all index subsets $K \subseteq \{1, 2, \ldots, N\}$, where $W_K = [W^i \mid i \in K] = \begin{bmatrix} W^{i_1} & W^{i_2} & \cdots & W^{i_{|K|}} \end{bmatrix}$. Since the left part of the modal controllability matrix $\mathcal{C}_{\rm m}(\bigoplus_{k\in K} A, (\bigoplus_{k\in K} B)(W_K)^*)$ is described as $\bigoplus_{k\in K}(A - \lambda I)$, the rank deficiency in the left part is at least $|K|$, the cardinality of K, when λ is an eigenvalue of A. Therefore, for the right part of the modal controllability matrix,

$$\text{rank } [(\bigoplus_{k\in K} B)(W_K)^*] \ge |K| \tag{3.23}$$

has to be satisfied in order for $\mathcal{C}_{\rm m}(\bigoplus_{k\in K} A, (\bigoplus_{k\in K} B)(W_K)^*)$ to be of row full-rank. For the inequality (3.23), we must have

$$\text{rank } (W_K)^* \ge |K|. \tag{3.24}$$

The Rado–Perfect theorem (see Remark 1 below), a duality theorem known in matroid theory, shows that (3.24) is true for all K if and only if there exists $\{j_1, j_2, \ldots, j_N\}$ that satisfies (3.21). Thus the lemma is proved. ■

Remark 1. A linear algebraic version of the Rado–Perfect theorem is stated here for the readers' convenience, whereas the original theorem (17;p. 98) is given in matroid terms.

Rado–Perfect Theorem (linear algebraic version): For a family of vectors $\{\boldsymbol{a}_j \mid j \in S\}$ indexed by a finite set S and a partition $\{S_k \mid k = 1, 2, \ldots, N\}$ of S into N disjoint blocks, i.e., $\bigcup_{k=1}^{N} S_k = S$, $S_k \cap S_{k'} = \emptyset$ for $k \ne k'$, we have

$$\begin{aligned} &\max\{|J| \mid J \subseteq S, \{\boldsymbol{a}_j \mid j \in J\} : \text{linearly independent}, \\ &\qquad |J \cap S_k| \le 1 \ (1 \le k \le N)\} \\ &= \min\{N - |K| + \text{ rank } \{\boldsymbol{a}_j \mid j \in \bigcup_{k\in K} S_k\} \mid K \subseteq \{1, 2, \ldots, N\}\}. \end{aligned}$$

□

The following lemma, giving a necessary condition for the controllability, is directly led from the discussion above.

Lemma 4. If a G-symmetric system (A, B) is controllable with a functioning pattern F, then, for each $\mu \in R(G)$ satisfying $a^\mu \neq 0$, there exists a set of N^μ indices $\{j_1^\mu, j_2^\mu, \ldots, j_{N^\mu}^\mu\}$ $(1 \leq j_k^\mu \leq b^\mu, 1 \leq k \leq N^\mu)$ such that

$$\text{rank} \begin{bmatrix} (\boldsymbol{w}_{j_1^\mu}^1)^* \\ (\boldsymbol{w}_{j_2^\mu}^2)^* \\ \vdots \\ (\boldsymbol{w}_{j_{N^\mu}^\mu}^{N^\mu})^* \end{bmatrix} = N^\mu, \tag{3.25}$$

where $\boldsymbol{w}_{j_k^\mu}^k$ $(1 \leq k \leq N^\mu)$ is the j_k^μ-th column of the matrix $W_{Mk}^\mu = (\boldsymbol{w}_j^k \mid 1 \leq j \leq b^\mu) \in \mathbf{C}^{f \times b^\mu}$ defined by

$$W_M^\mu = (W_{Mk}^\mu \mid 1 \leq k \leq N^\mu). \tag{3.26}$$

□

(Proof) Apply Lemma 3 to each subsystem $(\bigoplus_{k=1}^{N^\mu}, (\bigoplus_{k=1}^{N^\mu} B^\mu)(W_M^\mu)^*)$ corresponding to μ. ■

As the final step, we derive Theorem 1 from Lemma 4. By Lemma 4 there exists, for each μ, a set of indices $\{j_1^\mu, j_2^\mu, \ldots, j_{N^\mu}^\mu\}$ such that the matrix $W_S^\mu = \begin{bmatrix} \boldsymbol{w}_{j_1^\mu}^1 & \boldsymbol{w}_{j_2^\mu}^2 & \cdots & \boldsymbol{w}_{j_{N^\mu}^\mu}^{N^\mu} \end{bmatrix}$ is of rank N^μ. This implies that W_S^μ has at least N^μ nonzero rows. The covering condition (3.7) follows from this fact combined with the observation that the matrix W_S^μ has at most $\sum_{l, a_l^\mu \neq 0} |M_l|$ nonzero rows, which is shown below.

Let the matrix W defined in (3.11) be decomposed as

$$W = (W_l \mid 1 \leq l \leq L), \tag{3.27}$$

according to the decomposition of the space $\mathcal{U}$ in (3.10) into L disjoint orbits as $\mathcal{U} = \bigoplus_{l=1}^L \mathcal{U}_l$, compatibly with (3.2). Then the matrix W_l $(1 \leq l \leq L)$ contains at most $|P_l|$ nonzero rows, and so does the matrix W_l^μ, which is a basis of the invariant subspace corresponding to D^μ in the l-th orbit and is given as

$$W_l = (W_l^\mu \mid \mu \in R(G)), \quad 1 \leq l \leq L. \tag{3.28}$$

Similarly to (3.19), the matrix W_l^μ is divided into two parts as

$$W_l^\mu = \begin{pmatrix} W_{Ml}^\mu \\ W_{Nl}^\mu \end{pmatrix}, \tag{3.29}$$

where W_{Ml}^μ is the first f rows of W_l^μ, corresponding to the functioning matrix F. Then the matrix W_{Ml}^μ contains at most $|M_l|$ nonzero rows from (3.5). The matrix W_{Mk}^μ defined in (3.26) is also decomposed into L disjoint orbits as

$$W_{Mk}^\mu = (W_{Mkl}^\mu \mid 1 \leq l \leq L, a_l^\mu \neq 0) \in \mathbf{C}^{f \times b^\mu}, \tag{3.30}$$

where $W_{Mkl}^\mu \in \mathbf{C}^{f \times b_l^\mu}$. Since the matrix W_{Ml}^μ is given as $W_{Ml}^\mu = (W_{Mkl}^\mu \mid 1 \leq k \leq N^\mu) \in \mathbf{C}^{f \times b_l^\mu N^\mu}$, the matrix W_{Mkl}^μ contains at most $|M_l|$ nonzero rows as the matrix W_{Ml}^μ does, hence the matrix W_{Mk}^μ in (3.30) contains at most $\sum_{l, a_l^\mu \neq 0} |M_l|$ nonzero rows. The proof of Theorem 1 is completed.

Remark 2. Since (A, B) satisfies the symmetry constraint (2.4) if and only if A and B are block diagonalized as shown in (3.16), the genericity under the symmetry condition (2.4) is equivalent to the genericity of the distinct diagonal blocks A^μ and B^μ in (3.16) in the sense that all of their entries are independent parameters. It should be emphasized that the genericity has not been utilized in the discussion above to derive the condition (3.21), which does not depend on the numerical data of the matrices A and B, but depends on the symmetry of the system.

Remark 3. The converse of the statement of Lemma 3 is not always true. Consider a system (A, B) given as

$$A = [\alpha], \quad B = [\beta_1\, \beta_2], \quad N = 3, \quad W^* = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 1 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 0 \\ \hline 1 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right].$$

In this case, the condition (3.21) holds for $j_1 = 1, j_2 = 1, j_3 = 2$. However, $\tilde{B}$ is calculated as

$$\tilde{B} = \begin{bmatrix} \beta_1 & 0 & \beta_2 \\ 0 & \beta_1 & 0 \\ \beta_1 & 0 & \beta_2 \end{bmatrix}$$

and the system is generically uncontrollable.

In connection to Lemma 3, it is worth mentioning the following lemma, given in (15) as a sufficient condition for controllability.

Lemma 5. Consider a system $(\bar{A}, \bar{B})$ with the matrices $\bar{A}$ and $\bar{B}$ given as

$$\bar{A} = \bigoplus_{k=1}^{N} A = \begin{bmatrix} A & & & \\ & A & & \\ & & \ddots & \\ & & & A \end{bmatrix}, \quad \bar{B} = (\bigoplus_{k=1}^{N} B)W^* = \begin{bmatrix} B(W^1)^* \\ B(W^2)^* \\ \vdots \\ B(W^N)^* \end{bmatrix}, \tag{3.31}$$

where $A \in \mathbf{C}^{a\times a}, B \in \mathbf{C}^{a\times b}$, and $W = (W^k \mid 1 \le k \le N) \in \mathbf{C}^{f\times bN}$ with $W^k = (\boldsymbol{w}_j^k \mid 1 \le j \le b) \in \mathbf{C}^{f\times b}$. Suppose that A and B are fully dense generic matrices. Then, the system $(\bar{A}, \bar{B})$ is controllable if there exists an integer j $(1 \le j \le b)$ such that

$$\text{rank} \begin{bmatrix} (\boldsymbol{w}_j^1)^* \\ (\boldsymbol{w}_j^2)^* \\ \vdots \\ (\boldsymbol{w}_j^N)^* \end{bmatrix} = N. \tag{3.32}$$

□

The sufficient condition in Lemma 5 is a special case $j_1 = j_2 = \cdots = j_N = j$ of the necessary condition in Lemma 3. It should be noted that the genericity is assumed explicitly in Lemma 5, whereas it has not been utilized in the discussion of Lemma 3.

4. Examples

4.1 *Spherical Diamond System*

Consider a spherical diamond system as shown in Fig. 2. It is symmetric with respect to $\mathcal{D}_6$, of order 12, which is defined by

$$\mathcal{D}_6 = \{e, \rho, \rho^2, \ldots, \rho^5; \sigma, \sigma\rho, \ldots, \sigma\rho^5\}, \tag{4.1}$$

with $\rho^6 = \sigma^2 = (\sigma\rho)^2 = e$ (e is the identity element). The group $\mathcal{D}_6$ generally represents the geometric symmetry of a regular hexagon.

The index set of all the irreducible representations of $\mathcal{D}_6$, denoted as $R(\mathcal{D}_6)$, is given by

$$R(\mathcal{D}_6) = \{A_1, A_2, B_1, B_2, E_1, E_2\}, \tag{4.2}$$

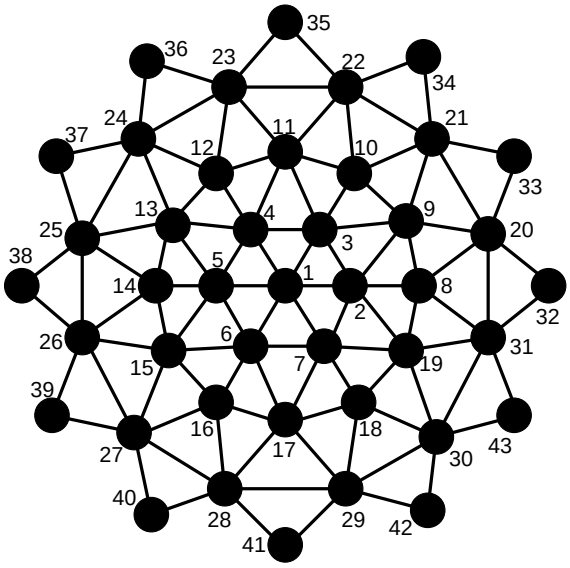

Figure 2.. $\mathcal{D}_6$-symmetric spherical diamond system consisting of identical $m = 43$ modules. figure

where A_1, A_2, B_1 and B_2 are one-dimensional irreducible matrix representations given as

$$T^{A_1}(g) = 1 \quad (g \in \mathcal{D}_6),$$

$$T^{A_2}(g) = \begin{cases} 1 \ (g \in \{\rho^i\}), \\ -1 \ (g \in \{\sigma\rho^i\}), \end{cases}$$

with $0 \leq i \leq 5$,

$$T^{B_1}(g) = \begin{cases} 1 \ (g \in \{\sigma\rho^{2j}, \rho^{2j}\}), \\ -1 \ (g \in \{\sigma\rho^{2j+1}, \rho^{2j+1}\}), \end{cases}$$

with $0 \leq j \leq 2$,

$$T^{B_2}(g) = \begin{cases} 1 \ (g \in \{\sigma\rho^{2j+1}, \rho^{2j}\}), \\ -1 \ (g \in \{\sigma\rho^{2j}, \rho^{2j+1}\}), \end{cases}$$

with $0 \leq j \leq 2$, and E_k $(k = 1, 2)$ are two-dimensional ones given as

$$T^{E_k}(\rho^j) = \begin{bmatrix} \cos kj\theta & -\sin kj\theta \\ \sin kj\theta & \cos kj\theta \end{bmatrix}, \quad T^{E_k}(\sigma\rho^j) = \begin{bmatrix} \cos kj\theta & -\sin kj\theta \\ -\sin kj\theta & -\cos kj\theta \end{bmatrix},$$

with $\theta = \frac{\pi}{3}$ and $0 \leq j \leq 5$.

The symmetry condition (2.4) holds for T and S defined naturally as representations of permutations. Note that $T = S$. The irreducible representation decomposition of S of $\mathcal{D}_6$ is described as

$$S = 7A_1 + A_2 + 4B_1 + 3B_2 + 7E_1 + 7E_2. \tag{4.3}$$

The system in Fig. 2 consists of 7 orbits with P_l $(1 \leq l \leq 7)$ given as

$$P_1 = \{1\}, \quad P_2 = \{2, 3, 4, 5, 6, 7\}, \quad P_3 = \{8, 10, 12, 14, 16, 18\}, \quad P_4 = \{9, 11, 13, 15, 17, 19\},$$
$$P_5 = \{20, 21, 22, \ldots, 31\}, \quad P_6 = \{32, 34, 36, 38, 40, 42\}, \quad P_7 = \{33, 35, 37, 39, 41, 43\}.$$

Corresponding to the decomposition into seven orbits as above, the permutation representations S_l $(1 \leq l \leq 7)$ are decomposed into irreducible representations as

$S_1 = A_1$, $S_2 = A_1 + B_1 + E_1 + E_2$, $S_3 = A_1 + B_1 + E_1 + E_2$,
$S_4 = A_1 + B_2 + E_1 + E_2$, $S_5 = A_1 + A_2 + B_1 + B_2 + 2E_1 + 2E_2$,
$S_6 = A_1 + B_1 + E_1 + E_2$, $S_7 = A_1 + B_2 + E_1 + E_2$.

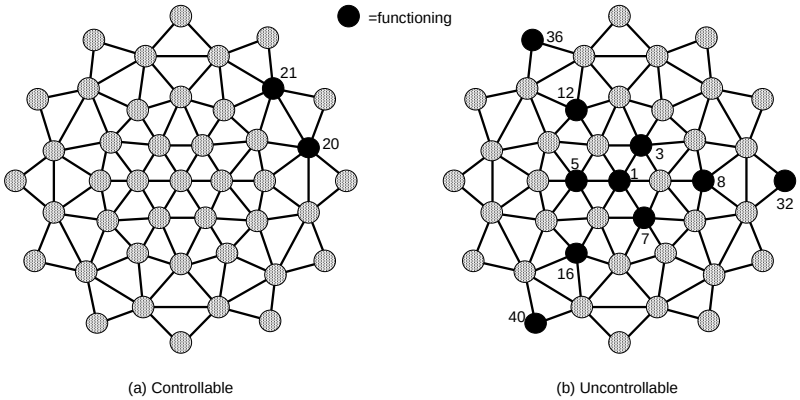

Figure 3.. A symmetric spherical diamond system with some functioning patterns.
figure

Suppose that the functioning modules are indexed by $M = \bigcup_{l=1}^{7} M_l$. Then, the condition (3.7) reads

$$\begin{array}{lccccccccccc}
A_1: & |M_1| & + & |M_2| & + & |M_3| & + & |M_4| & + & |M_5| & + & |M_6| & + & |M_7| \geq 1, \\
A_2: & & & & & & & & & |M_5| & & & & \geq 1, \\
B_1: & & & |M_2| & + & |M_3| & & & + & |M_5| & + & |M_6| & & \geq 1, \\
B_2: & & & & & & & |M_4| & + & |M_5| & & & + & |M_7| \geq 1, \\
E_1: & & & |M_2| & + & |M_3| & + & |M_4| & + & |M_5| & + & |M_6| & + & |M_7| \geq 2, \\
E_2: & & & |M_2| & + & |M_3| & + & |M_4| & + & |M_5| & + & |M_6| & + & |M_7| \geq 2.
\end{array}$$

This system of inequalities is equivalent to

$$|M_5| \geq 1, \quad |M_2| + |M_3| + |M_4| + |M_5| + |M_6| + |M_7| \geq 2. \tag{4.4}$$

Therefore,

$$|M| = \sum_{l=1}^{7} |M_l| \geq 2. \tag{4.5}$$

In this example, the lower bound for $|M|$ obtained from the covering condition in Theorem 1 coincides with $\max_{\mu \in R(G)} N^{\mu} = 2$ in Corollary 1. This lower bound is attained e.g., by $M = \{20, 21\}$ (Fig. 3(a)).

For the system shown in Fig. 3(b), for example, the indices of the functioning modules M_l $(1 \leq l \leq 7)$ are given as

$$\begin{aligned}
&M_1 = \{1\}, \quad M_2 = \{3, 5, 7\}, \quad M_3 = \{8, 12, 16\}, \quad M_4 = \emptyset, \\
&M_5 = \emptyset, \quad M_6 = \{32, 36, 40\}, \quad M_7 = \emptyset.
\end{aligned}$$

The covering condition is not satisfied for A_2 and B_2. Therefore, from Theorem 1, the system with this functioning pattern is revealed to be uncontrollable, although it has more inputs than the lower bound.

4.2 *Cubic System*

Consider a cubic system with a regular octahedron inside it (Fig. 1). The system is symmetric with respect to the group $\mathcal{O}_h$ of order 48, the symmetry group of the cube. The group is obtained as the direct product of the octahedral group $\mathcal{O}$ of order 24 and the group of reflection $\mathcal{Z}_2 = \{e, \sigma\}$,

where $\sigma^2 = e$. The 24 elements of $\mathcal{O}$ are classified into five conjugacy classes:

$\mathcal{E} = \{e\}$,
$\mathcal{C}_4^2 = \{$three rotations of π about fourfold axes$\}$,
$\mathcal{C}_2 = \{$six rotations of π about twofold axes$\}$,
$\mathcal{C}_4 = \{$three rotations of $\pi/2$ and three rotations of $\frac{3\pi}{2}$ about fourfold axes$\}$,
$\mathcal{C}_3 = \{$four rotations of $\frac{2\pi}{3}$ and four rotations of $\frac{4\pi}{3}$ about threefold axes$\}$,

according to the notation of (9). The index set $R(\mathcal{O})$ of the irreducible representations of $\mathcal{O}$ is given by

$$R(\mathcal{O}) = \{A_1, A_2, E, T_1, T_2\},$$

where A_1 is the one-dimensional unit representation, A_2 is a non-unit one-dimensional representation, E is a two-dimensional irreducible representation, and T_1 and T_2 are three-dimensional ones. The products of elements in $\mathcal{O}$ with those of $\mathcal{Z}_2$ produces the 48 elements of $\mathcal{O}_h$. The index set $R(\mathcal{O}_h)$ of the irreducible representations is then given by

$$R(\mathcal{O}_h) = \{A_1, A_2, E, T_1, T_2, A_1', A_2', E', T_1', T_2'\}.$$

The character table of the octahedral group $\mathcal{O}_h$ is given in Table 1.

The system is symmetric with respect to the group $\mathcal{O}_h$ with two orbits $P_1 = \{1,2,3,4,5,6,7,8\}$ and $P_2 = \{9,10,11,12,13,14\}$. The system matrices A and B for the system (Fig. 1) are given as

$$A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right], \quad B = \left[\begin{array}{c|c} B_{11} & O \\ \hline O & B_{22} \end{array}\right], \tag{4.6}$$

where

$$A_{11} = \begin{bmatrix} \alpha_1 & \beta_1 & 0 & \beta_1 & \beta_1 & 0 & 0 & 0 \\ \beta_1 & \alpha_1 & \beta_1 & 0 & 0 & \beta_1 & 0 & 0 \\ 0 & \beta_1 & \alpha_1 & \beta_1 & 0 & 0 & \beta_1 & 0 \\ \beta_1 & 0 & \beta_1 & \alpha_1 & 0 & 0 & 0 & \beta_1 \\ \beta_1 & 0 & 0 & 0 & \alpha_1 & \beta_1 & 0 & \beta_1 \\ 0 & \beta_1 & 0 & 0 & \beta_1 & \alpha_1 & \beta_1 & 0 \\ 0 & 0 & \beta_1 & 0 & 0 & \beta_1 & \alpha_1 & \beta_1 \\ 0 & 0 & 0 & \beta_1 & \beta_1 & 0 & \beta_1 & \alpha_1 \end{bmatrix}, \quad B_{11} = \begin{bmatrix} \gamma_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \gamma_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \gamma_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma_1 \end{bmatrix}, \tag{4.7}$$

$$A_{22} = \begin{bmatrix} \alpha_2 & \beta_2 & \beta_2 & \beta_2 & \beta_2 & 0 \\ \beta_2 & \alpha_2 & \beta_2 & 0 & \beta_2 & \beta_2 \\ \beta_2 & \beta_2 & \alpha_2 & \beta_2 & 0 & \beta_2 \\ \beta_2 & 0 & \beta_2 & \alpha_2 & \beta_2 & \beta_2 \\ \beta_2 & \beta_2 & 0 & \beta_2 & \alpha_2 & \beta_2 \\ 0 & \beta_2 & \beta_2 & \beta_2 & \beta_2 & \alpha_2 \end{bmatrix}, \quad B_{22} = \begin{bmatrix} \gamma_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \gamma_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma_2 \end{bmatrix}. \tag{4.8}$$

TABLE 1. The character table of the octahedral group $\mathcal{O}_h$.

table

	$\mathcal{E}$	$\mathcal{C}_4^2$	$\mathcal{C}_2$	$\mathcal{C}_4$	$\mathcal{C}_3$	$\mathcal{E}'$	$\mathcal{C}_4^{2\prime}$	$\mathcal{C}_2{}'$	$\mathcal{C}_4{}'$	$\mathcal{C}_3{}'$
A_1	1	1	1	1	1	1	1	1	1	1
A_2	1	1	−1	−1	1	1	1	−1	−1	1
E	2	2	0	0	−1	2	2	0	0	−1
T_1	3	−1	−1	1	0	3	−1	−1	1	0
T_2	3	−1	1	−1	0	3	−1	1	−1	0
A_1'	1	1	1	1	1	−1	−1	−1	−1	−1
A_2'	1	1	−1	−1	1	−1	−1	1	1	−1
E'	2	2	0	0	−1	−2	−2	0	0	1
T_1'	3	−1	−1	1	0	−3	1	1	−1	0
T_2'	3	−1	1	−1	0	−3	1	−1	1	0

The symmetry condition (2.4) holds for T and S defined naturally as representations of permutations. Note that $T = S$. The corresponding permutation representations S_l ($l \in \{1,2\}$) are decomposed into irreducible representations as

$$S_1 = A_1 + T_2 + A_1' + T_2', \quad S_2 = A_1 + E + T_2'. \tag{4.9}$$

Suppose that the functioning modules are indexed by $M = \bigcup_{l=1}^{2} M_l$. Then, the covering condition (3.7) reads

$$\begin{array}{rl} A_1: & |M_1| + |M_2| \geq 1, \\ E: & |M_2| \geq 2, \\ T_2: & |M_1| \geq 3, \\ A_1': & |M_1| \geq 1, \\ T_2': & |M_1| + |M_2| \geq 3. \end{array}$$

This system of inequalities is equivalent to

$$|M_1| \geq 3, \quad |M_2| \geq 2.$$

Therefore,

$$|M| = \sum_{l=1}^{2} |M_l| \geq 5.$$

In this case, the lower bound for $|M|$ obtained by the covering condition in Theorem 1 is greater than $\max_{\mu \in R(G)} N^{\mu} = 3$ in Corollary 1.

For example, the system with M_l ($l = 1, 2$) given as

$$M_1 = \{3, 6, 8\}, \quad M_2 = \{10, 11\}$$

satisfies the covering condition (and is in fact generically controllable). Thus, the lower bound obtained above from the covering condition is attained in this example.

Another example for M_l ($l = 1, 2$) given as

$$M_1 = \{3, 5\}, \quad M_2 = \{9, 14\}$$

does not satisfy the covering condition for T_2. According to Theorem 1, the system is uncontrollable.

5. Conclusion

The present paper has carried out a quantitative analysis on the controllability of symmetric systems. We have shown a lower bound for the number of functioning modules needed to keep the entire symmetric system controllable. The covering condition has been derived based on the orbit decomposition of the symmetric system. Since our concern has been the characteristics derived from the symmetry of the systems, we have discussed the lower bound merely from the symmetry of the structures, and not from the numerical information nor from the sparsity of the connections among the modules. Generally, if structurally independent modules are connected, the system keeps its controllability with only one functioning module. However, for symmetric systems, we have shown the cases where more than one functioning module is needed to keep the entire system controllable. Moreover, the lower bound obtained by the covering condition is verified to be tight in our numerical examples.

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Appendix: Preliminaries on Group Representation

We describe some fundamental facts about group representations (see, for example (9)) utilized in the discussion above.

Let G be a finite group. The *order* of G is the number of elements in G. Let V be a finite-dimensional vector space over $\mathbf{C}$, where $\mathbf{C}$ is the field of complex numbers, and denote by $GL(V)$ the group of all nonsingular linear transformations of V onto itself. A *representation* of G on *representation space* V is a homomorphism $\tau : G \to GL(V)$. The *dimension* of the representation is $N = \dim V$. A subspace W of V is *invariant* under τ if $\tau(g)\boldsymbol{w} \in W$ for every $g \in G, \boldsymbol{w} \in W$. A representation τ is *irreducible* if the only invariant subspaces of V are $\{0\}$ and V itself. An n-dimensional matrix representation of G is a homomorphism $T : G \to GL(n)$, where $GL(n)$ denotes the group of all nonsingular matrices over $\mathbf{C}$ of order n. If a basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n\}$ is fixed for V, we obtain a matrix representation T of G from τ. We say that T is *unitary* if $T(g)$ is unitary for each $g \in G$. Any matrix representation T is equivalent to a unitary matrix representation, in the sense that two matrix representations T_1 and T_2 of G are *equivalent* if there exists a nonsingular matrix K such that

$$T_1(g) = K^{-1}T_2(g)K, \quad g \in G. \tag{1}$$

We denote by $\{\tau^\mu \mid \mu \in R(G)\}$ a complete list of nonequivalent irreducible representations of G, where $R(G)$ denotes an index set for the irreducible representations of G. A complete list of nonequivalent irreducible matrix representations of G is denoted by $\{D^\mu \mid \mu \in R(G)\}$. We denote the dimension of D^μ by N^μ. Every finite-dimensional representation of a finite group can be decomposed into a direct sum of irreducible representations. The direct sum decomposition is obtained by

$$V = \bigoplus_{\mu \in R(G)} \bigoplus_{i=1}^{a^\mu} V_i^\mu,$$

where V_i^μ are invariant subspaces of V which transform irreducibly under the restrictions of τ to V_i^μ, and a^μ is called the *multiplicity* of τ^μ in τ. Then the matrix representation can be put into a block-diagonal form

$$T(g) = \bigoplus_{\mu \in R(G)} \bigoplus_{i=1}^{a^\mu} D^\mu(g), \quad g \in G, \tag{2}$$

where D^μ is irreducible, if we first choose an adequate basis $\{\boldsymbol{v}_{ij}^\mu \mid j = 1, \cdots, N^\mu\}$ for each V_i^μ and adopt their union as a basis of V. We write (2) as $T = \sum_{\mu \in R(G)} a^\mu \mu$. The following is known as Schur's lemma.

Lemma A.1.
Let D^1 and D^2 be irreducible matrix representations of G over $\mathbf{C}$, and H be a matrix over $\mathbf{C}$. Assume that

$$D^1(g)H = HD^2(g), \qquad g \in G.$$

1. If D^1 and D^2 are not equivalent, then $H = O$.
2. If $D^1 = D^2$, $g \in G$, and D^1 is irreducible, then $H = \alpha I$ for some $\alpha \in \mathbf{C}$.

□

By Schur's lemma, the block diagonalizatoin of the matrix A as shown in (3.16) is carried out as follows. The equation (2.4) is rewritten as

$$(Z^*T(g)Z)(Z^*AZ) = (Z^*AZ)(Z^*T(g)Z), \quad g \in G, \tag{3}$$

and thus, from the equation (3.15),

$$(\bigoplus_{\mu\in R(G)}\bigoplus_{i=1}^{a^\mu} D^\mu(g))\tilde{A} = \tilde{A}(\bigoplus_{\mu\in R(G)}\bigoplus_{i=1}^{a^\mu} D^\mu(g)), \quad g\in G, \tag{4}$$

with $\tilde{A} = Z^*AZ = ((Z^\mu)^*AZ^{\mu'} \mid \mu,\mu' \in R(G))$. From the first statement of Schur's lemma,

$$(Z^\mu)^*AZ^{\mu'} = O, \quad \text{if } \mu \neq \mu'. \tag{5}$$

Therefore,

$$\tilde{A} = \bigoplus_{\mu\in R(G)} \bar{A}^\mu, \tag{6}$$

where $\bar{A}^\mu = (Z^\mu)^*AZ^\mu \in \mathbf{C}^{(a^\mu N^\mu)\times(a^\mu N^\mu)}$. Moreover, from the second statement of Schur's lemma,

$$(Z_i^\mu)^*AZ_j^\mu = \alpha_{ij}^\mu I_{N^\mu}, \tag{7}$$

with $1 \le i,j \le a^\mu$ for some $\alpha_{ij} \in \mathbf{C}$. Therefore, the matrix $\bar{A}^\mu$ is equal, up to permutations of rows and columns, to a block-diagonal matrix consisting of N^μ copies of an identical matrix $A^\mu = (\alpha_{ij}^\mu)$ of size $a^\mu \times a^\mu$. To sum up,

$$\tilde{A} = \bigoplus_{\mu\in R(G)}\bigoplus_{k=1}^{N^\mu} A^\mu \tag{8}$$

is derived as in (3.16). Similar calculation leads to the block diagonalization of B as in (3.16).

A *permutation* of a nonempty set X is a 1-1 mapping of X onto itself. The set S_X of all permutations of X forms a group, the *full symmetric group* on X. Elements of S_X are said to *act* or *operate* on elements of X. Two elements x and y ($x, y \in X$) are said to belong to the same orbit if and only if $y = \tau(g)(x)$ for some $g \in G$. The orbit containing x is the set $\{\tau(g)x \mid g \in G\}$. If there is only one orbit in X we say G is transitive.

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