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On the Relationship between L-convex Functions and Submodular Integrally Convex Functions

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Abstract

This paper shows the equivalence between Murota's L-convex functions and Favati and Tardella's submodular integrally convex functions: For a submodular integrally convex function $g(p_1, \ldots, p_n)$, the function $\tilde{g}$ defined by $\tilde{g}(p_0, p_1, \ldots, p_n) = g(p_1 - p_0, \ldots, p_n - p_0)$ is an L-convex function, and vice versa. This fact implies, in combination with known results for L-convex functions, that submodular integrally convex functions enjoy a discrete separation property and that they are characterized as the Fenchel-Legendre conjugates of M-convex functions on generalized polymatroids.

Keywords: integral convexity, submodularity, conjugacy.
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1 Introduction

Convexity for discrete functions has been a continual research topic. Among others, Miller [13] was a forerunner in the early seventies; in the early eighties, Frank [5], Fujishige [8, 9], Lovász [12] investigated submodular set functions in terms of discrete convexity, while Kindler [11] gave a general setting for discrete separation theorems.

In [4] Favati and Tardella have introduced a class of discrete functions (integer-valued functions defined on integer lattice points) called integrally convex functions. For a function $g : \mathbf{Z}^n \to \mathbf{Z} \cup \{+\infty\}$ its piecewise-convex extension $\overline{g} : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ is defined by

$$\overline{g}(b) = \min\{ \sum_{p \in \mathrm{N}(b)} \lambda_p g(p) \mid \sum_{p \in \mathrm{N}(b)} \lambda_p p = b, \sum_{p \in \mathrm{N}(b)} \lambda_p = 1, \lambda_p \geq 0 \; (p \in \mathrm{N}(b)) \}, \quad b \in \mathbf{R}^n,$$

where

$$\mathrm{N}(b) = \{ p \in \mathbf{Z}^n \mid \lfloor b_i \rfloor \leq p_i \leq \lceil b_i \rceil \; (i = 1, \ldots, n) \}, \quad b \in \mathbf{R}^n,$$

denotes the set of the vertices of the smallest rectangle that contains $b \in \mathbf{R}^n$ in its convex hull. Here it should be clear that $\lfloor t \rfloor = \max\{a \in \mathbf{Z} \mid a \leq t\}$ and $\lceil t \rceil = \min\{a \in \mathbf{Z} \mid a \geq t\}$ for $t \in \mathbf{R}$. By construction, $\overline{g}$ is convex in each unit hypercube and $\overline{g}(p) = g(p)$ for $p \in \mathbf{Z}^n$. A function $g : \mathbf{Z}^n \to \mathbf{Z} \cup \{+\infty\}$ is said to be integrally convex if its piecewise-convex extension $\overline{g} : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ is (globally) convex. In the original definition, the effective domain of g:

$$\mathrm{dom}\, g = \{ p \in \mathbf{Z}^n \mid g(p) < +\infty \}$$

of an integrally convex function g is assumed to be a discrete rectangle ("box"). In this paper, however, we allow $\mathrm{dom}\, g$ to be a more general nonempty set. A function $g : \mathbf{Z}^n \to \mathbf{Z} \cup \{+\infty\}$ with $\mathrm{dom}\, g \neq \emptyset$ is said to be submodular if

$$g(p) + g(q) \geq g(p \vee q) + g(p \wedge q), \qquad p, q \in \mathbf{Z}^n,$$

where $p \vee q$ and $p \wedge q$ are, respectively, the vectors of componentwise maxima and minima, i.e.,

$$(p \vee q)_i = \max(p_i, q_i), \quad (p \wedge q)_i = \min(p_i, q_i), \qquad i = 1, \ldots, n.$$

See Edmonds [3], Frank–Tardos [7], Fujishige [10], Lovász [12], and Topkis [21] for general background on submodular functions.

It is shown in Favati–Tardella [4] that a global minimum of an integrally convex function g over a discrete rectangle can be characterized as a local minimum, i.e., for $p \in \mathrm{dom}\, g$, we have $g(p) \leq g(q)$ for all $q \in \mathbf{Z}^n$ if and only if $g(p) \leq g(q)$ for all $q \in \mathbf{Z}^n$ with $||p - q||_\infty = 1$, where

$$||p - q||_\infty = \max_{1 \leq i \leq n} |p_i - q_i|.$$

It is also shown in Favati–Tardella [4] that a submodular integrally convex function over a discrete rectangle can be minimized in polynomial time.

Furthermore, a succinct characterization of submodular integrally convex functions over a discrete rectangle has been obtained in Favati–Tardella [4]

Theorem 1.1 (Favati–Tardella [4, Cor.5.2.2]) *A submodular function $g : \mathbf{Z}^n \to \mathbf{Z} \cup \{+\infty\}$ with a nonempty discrete rectangle effective domain is integrally convex if and only if*

$$g(p) + g(q) \geq g\left(\left\lceil \frac{p+q}{2} \right\rceil\right) + g\left(\left\lfloor \frac{p+q}{2} \right\rfloor\right), \quad ||p - q||_\infty = 2,\ p, q \in \mathbf{Z}^n.$$

□

It should be clear that $\lceil p \rceil = (\lceil p_i \rceil \mid i = 1, \ldots, n) \in \mathbf{Z}^n$ and $\lfloor p \rfloor = (\lfloor p_i \rfloor \mid i = 1, \ldots, n) \in \mathbf{Z}^n$ for $p = (p_i \mid i = 1, \ldots, n) \in \mathbf{R}^n$. Note that the above theorem implies the following as an immediate corollary.

Corollary 1.2 *A function $g : \mathbf{Z}^n \to \mathbf{Z} \cup \{+\infty\}$ with a nonempty discrete rectangle effective domain is submodular and integrally convex if and only if*

$$g(p) + g(q) \geq g\left(\left\lceil \frac{p+q}{2} \right\rceil\right) + g\left(\left\lfloor \frac{p+q}{2} \right\rfloor\right), \quad ||p - q||_\infty \leq 2,\ p, q \in \mathbf{Z}^n.$$

□

Another class of discrete functions, called L-convex functions, has been introduced by Murota [17] as a generalization of the Lovász extension of submodular set functions. A function $\tilde{g} : \mathbf{Z}^{\tilde{n}} \to \mathbf{Z} \cup \{+\infty\}$ with $\mathrm{dom}\, \tilde{g} \neq \emptyset$ is said to be L-convex if

$$\tilde{g}(\tilde{p}) + \tilde{g}(\tilde{q}) \geq \tilde{g}(\tilde{p} \vee \tilde{q}) + \tilde{g}(\tilde{p} \wedge \tilde{q}), \qquad \tilde{p}, \tilde{q} \in \mathbf{Z}^{\tilde{n}}, \tag{1.1}$$

$$\exists r \in \mathbf{Z}, \forall \tilde{p} \in \mathbf{Z}^{\tilde{n}} :\ \tilde{g}(\tilde{p} + \tilde{\mathbf{1}}) = \tilde{g}(\tilde{p}) + r, \tag{1.2}$$

where $\tilde{\mathbf{1}} = (1, 1, \ldots, 1) \in \mathbf{Z}^{\tilde{n}}$. It can be shown [17] that the Lovász extension of an integer-valued submodular set function (Lovász [12]; see also Fujishige [9, 10]) is nothing but an L-convex function that has an additional property of positive homogeneity:

$$\tilde{g}(\alpha\tilde{p}) = \alpha\tilde{g}(\tilde{p}), \qquad 0 < \alpha \in \mathbf{Z}, \tilde{p} \in \mathbf{Z}^{\tilde{n}}.$$

The second condition in the definition of L-convexity means the linearity of $\tilde{g}$ in the direction of $\tilde{\mathbf{1}}$. In this paper we assume $r = 0$, a kind of normalization, unless otherwise stated. That is, we shall say $\tilde{g} : \mathbf{Z}^{\tilde{n}} \to \mathbf{Z} \cup \{+\infty\}$ with $\mathrm{dom}\,\tilde{g} \neq \emptyset$ is L-convex if it is submodular and

$$\tilde{g}(\tilde{p} + \tilde{\mathbf{1}}) = \tilde{g}(\tilde{p}), \qquad \tilde{p} \in \mathbf{Z}^{\tilde{n}}. \tag{1.3}$$

The following fact is observed in [18] on the basis of an argument in [17]; the proof is given in §4.

Theorem 1.3 (Murota [18]) *An L-convex function (with any r in (1.2)) is submodular and integrally convex.* □

It is shown in [17] that L-convex functions are in one-to-one correspondence with M-convex functions, yet another class of discrete functions introduced in Murota [16] as a generalization of valuated matroids due to Dress–Wenzel [1, 2] (see §3.1 for M-convex functions and this correspondence). This correspondence is based on a discrete analogue of the Fenchel–Legendre transformation in convex analysis (Rockafellar [20]) and generalizes a fundamental fact in matroid theory (Welsh [22], White [23]) that the submodularity of the rank function is cryptomorphically equivalent to the exchange axiom for independent sets. Furthermore, a discrete separation theorem has been established for L-convex functions as well as for M-convex functions (see §3.2 for separation theorems).

This paper reveals the equivalence between L-convex functions and submodular integrally convex functions: For an L-convex function $\tilde{g}(p_0, p_1, \ldots, p_n)$, the function g defined by $g(p_1, \ldots, p_n) = \tilde{g}(0, p_1, \ldots, p_n)$ is a submodular integrally convex function, and conversely, for a submodular integrally convex function $g(p_1, \ldots, p_n)$, the function $\tilde{g}$ defined by

$$\tilde{g}(p_0, p_1, \ldots, p_n) = g(p_1 - p_0, \ldots, p_n - p_0) \tag{1.4}$$

is an L-convex function. This fact implies, in combination with known results for L-convex functions and M-convex functions, that a discrete separation theorem holds for submodular integrally convex functions and that submodular integrally convex functions are the Fenchel–Legendre conjugates of M-convex functions on generalized polymatroids considered in Murota–Shioura [19]. The results are described in §2 with their proofs in §4. The implications of the present results are discussed in §3.

2 Results

The main result of this paper is the following.

Theorem 2.1 *A function $g : \mathbf{Z}^n \to \mathbf{Z} \cup \{+\infty\}$ is submodular and integrally convex if and only if $g(p_1, \ldots, p_n) = \tilde{g}(0, p_1, \ldots, p_n)$ for some L-convex function $\tilde{g} : \mathbf{Z}^{\tilde{n}} \to \mathbf{Z} \cup \{+\infty\}$ with $\tilde{n} = n + 1$. Such an L-convex function $\tilde{g}$ is uniquely determined by g as $\tilde{g}(p_0, p_1, \ldots, p_n) = g(p_1 - p_0, \ldots, p_n - p_0)$.* $\square$

This theorem is proven on the basis of the following theorem, which includes some properties not needed for the proof but worth mentioning in their own right. The proofs are given in §4.

We put $V = \{1, \ldots, n\}$ and denote the characteristic vector of $X \subseteq V$ by $\chi_X \in \{0, 1\}^V$.

Theorem 2.2 *For $g : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ the following conditions,* (a) *to* (f)*, are equivalent:*
(a) *Submodularity* (SBM)*:*

$$g(p) + g(q) \geq g(p \vee q) + g(p \wedge q), \quad p, q \in \mathbf{Z}^V \tag{2.1}$$

& Integral convexity (IC)*: g is integrally convex;*
(b) *Discrete midpoint convexity* (MPC)*:*

$$g(p) + g(q) \geq g\left(\left\lceil \frac{p+q}{2} \right\rceil\right) + g\left(\left\lfloor \frac{p+q}{2} \right\rfloor\right), \quad p, q \in \mathbf{Z}^V; \tag{2.2}$$

(c) *Local discrete midpoint convexity* (L-MPC)*:*

$$g(p) + g(q) \geq g\left(\left\lceil \frac{p+q}{2} \right\rceil\right) + g\left(\left\lfloor \frac{p+q}{2} \right\rfloor\right), \quad ||p - q||_\infty \leq 2,\ p, q \in \mathbf{Z}^V \tag{2.3}$$

& Domain condition (DOM)*:*

$$p, q \in \mathrm{dom}\, g,\ 0 \leq \alpha \in \mathbf{Z} \quad \Longrightarrow \quad (p - \alpha\mathbf{1}) \vee q,\ p \wedge (q + \alpha\mathbf{1}) \in \mathrm{dom}\, g; \tag{2.4}$$

(d) *Local submodularity* (L-SBM)*:*

$$g(p + \chi_X) + g(p + \chi_Y) \geq g(p + \chi_{X \cup Y}) + g(p + \chi_{X \cap Y}), \quad X, Y \subseteq V,\ p \in \mathbf{Z}^V \tag{2.5}$$

& Local projected submodularity (L-PR-SBM)*:*

$$g(p + \chi_X + \mathbf{1}) + g(p + \chi_Y) \geq g(p + \chi_{X \cup Y}) + g(p + \chi_{X \cap Y} + \mathbf{1}), \quad X, Y \subseteq V,\ p \in \mathbf{Z}^V \tag{2.6}$$

& DOM (2.4);

(e) *Projected submodularity* (PR-SBM)*:*

$$g(p) + g(q) \geq g((p - \alpha\mathbf{1}) \vee q) + g(p \wedge (q + \alpha\mathbf{1})), \quad 0 \leq \alpha \in \mathbf{Z},\ p, q \in \mathbf{Z}^V; \tag{2.7}$$

(f) *Unit projected submodularity* (UPR-SBM)*:*

$$g(p) + g(q) \geq g((p - \mathbf{1}) \vee q) + g(p \wedge (q + \mathbf{1})), \quad p, q \in \mathbf{Z}^V \tag{2.8}$$

& SBM (2.1). $\square$

Remark 2.1 As already mentioned in Corollary 1.2, the equivalence between (a) and (c) for g with a discrete rectangle effective domain has been shown by Favati–Tardella [4]. $\square$

Remark 2.2 For g with a general $\mathrm{dom}\, g$, the local characterizations in Theorem 1.1 and Corollary 1.2 are valid only under a certain assumption, like (2.4), on $\mathrm{dom}\, g$. Consider, for example, $g : \mathbf{Z}^2 \to \mathbf{Z} \cup \{+\infty\}$ with $\mathrm{dom}\, g = \{(3, 0), (0, 3)\}$. This function vacuously satisfies the local discrete midpoint convexity (2.3), but is not a submodular integrally convex function. Because of this possible technical complication we shall provide in §4 self-contained proofs without relying on an innocent extension of Corollary 1.2, though it turns out to be correct. $\square$

Remark 2.3 The domain condition DOM (2.4) is equivalent to a seemingly weaker condition:

$$p, q \in \mathrm{dom}\, g,\ \alpha \in \{0, 1\} \quad \Longrightarrow \quad (p - \alpha\mathbf{1}) \vee q,\ p \wedge (q + \alpha\mathbf{1}) \in \mathrm{dom}\, g. \tag{2.9}$$

$\square$

3 Implications

3.1 Conjugacy

L-convex functions are known to be the Fenchel–Legendre conjugates of M-convex functions. In this section we explain this result and point out important implications of the present result in relation to conjugacy.

Put $\tilde{V} = \{0, 1, \ldots, n\}$. A function $\tilde{f} : \mathbf{Z}^{\tilde{V}} \to \mathbf{Z} \cup \{+\infty\}$ with $\mathrm{dom}\, \tilde{f} \neq \emptyset$ is said to be M-convex if the following variant of the simultaneous exchange axiom holds true:

(M-EXC) For $\tilde{x}, \tilde{y} \in \mathrm{dom}\, \tilde{f}$ and $i \in \mathrm{supp}^+(\tilde{x} - \tilde{y})$ there exists $j \in \mathrm{supp}^-(\tilde{x} - \tilde{y})$ such that

$$\tilde{f}(\tilde{x}) + \tilde{f}(\tilde{y}) \geq \tilde{f}(\tilde{x} - \chi_i + \chi_j) + \tilde{f}(\tilde{y} + \chi_i - \chi_j).$$

where χ_i is the characteristic vector of $i \in \tilde{V}$, and we denote the positive support and the negative support of $\tilde{x} = (\tilde{x}_i \mid i \in \tilde{V}) \in \mathbf{Z}^{\tilde{V}}$ by

$$\mathrm{supp}^+(\tilde{x}) = \{i \in \tilde{V} \mid \tilde{x}_i > 0\}, \qquad \mathrm{supp}^-(\tilde{x}) = \{i \in \tilde{V} \mid \tilde{x}_i < 0\}.$$

The concept of M-convex function is a quantitative generalization of that of integral base set (=the set of integer points in the base polyhedron of an integral submodular system; see [10] for submodular systems). Note that $\mathrm{dom}\, \tilde{f}$ is an integral base set if $\tilde{f}$ is M-convex. When $\mathrm{dom}\, \tilde{f} \subseteq \{0, 1\}^{\tilde{V}}$, $\tilde{f}$ is M-convex if and only if $-\tilde{f}$ is a matroid valuation in the sense of Dress–Wenzel [1, 2].

For a function $\tilde{f} : \mathbf{Z}^{\tilde{V}} \to \mathbf{Z} \cup \{+\infty\}$ in general, we define its (integral Fenchel–Legendre) conjugate $\tilde{f}^{\bullet} : \mathbf{Z}^{\tilde{V}} \to \mathbf{Z} \cup \{+\infty\}$ by

$$\tilde{f}^{\bullet}(\tilde{p}) = \sup\{\langle \tilde{p}, \tilde{x} \rangle - \tilde{f}(\tilde{x}) \mid \tilde{x} \in \mathbf{Z}^{\tilde{V}}\}, \qquad \tilde{p} \in \mathbf{Z}^{\tilde{V}}, \tag{3.1}$$

where $\langle \tilde{p}, \tilde{x} \rangle = \sum_{i \in \tilde{V}} \tilde{p}_i \, \tilde{x}_i$. The mapping $\tilde{f} \mapsto \tilde{f}^{\bullet}$ is called the (convex) integral Fenchel–Legendre transformation.

Theorem 3.1 (Murota [17]) *The class of L-convex functions $\mathcal{L}$ (without the normalization of $r = 0$ in* (1.2)*) and that of M-convex functions $\mathcal{M}$ are in one-to-one correspondence under the integral Fenchel–Legendre transformation* (3.1)*. That is, for $\tilde{g} \in \mathcal{L}$ and $\tilde{f} \in \mathcal{M}$ we have $\tilde{g}^{\bullet} \in \mathcal{M}$, $\tilde{f}^{\bullet} \in \mathcal{L}$, $\tilde{g}^{\bullet\bullet} = \tilde{g}$, and $\tilde{f}^{\bullet\bullet} = \tilde{f}$.* □

It is known (cf. [10, Theorem 3.58]) that a generalized polymatroid of Frank [6] (see also Frank–Tardos [7]) is the projection of a base polyhedron along an axis.

Motivated by this fact, Murota–Shioura [19] considered M-convex functions on generalized polymatroids, which we name here M$^\natural$-convex functions. By definition, a function $f : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$, where $V = \{1, \ldots, n\}$, is M$^\natural$-convex if the function $\tilde{f} : \mathbf{Z}^{\tilde{V}} \to \mathbf{Z} \cup \{+\infty\}$ defined by

$$\tilde{f}(x_0, x) = \begin{cases} f(x) & (x_0 = -x(V)) \\ +\infty & (\text{otherwise}) \end{cases} \tag{3.2}$$

is M-convex, where $x(V) = \sum_{i \in V} x_i$. A translation of exchange axiom (M-EXC) for $\tilde{f}$ results in an axiom for an M$^\natural$-convex function f (see Murota–Shioura [19]):

(M$^\natural$-EXC) For $x, y \in \mathrm{dom}\, f$ and $i \in \mathrm{supp}^+(x - y)$,

$$\begin{aligned} f(x) + f(y) \;\; \geq \;\; \min \Bigl[& f(x - \chi_i) + f(y + \chi_i), \\ & \min_{j \in \mathrm{supp}^-(x-y)} \{ f(x - \chi_i + \chi_j) + f(y + \chi_i - \chi_j) \} \Bigr]. \end{aligned}$$

For symmetry of terminology, we say a function $g : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ with $\mathrm{dom}\, g \neq \emptyset$ is L$^\natural$-convex if $g(p) = \tilde{g}(0, p)$ (cf. (1.4)) for some L-convex function $\tilde{g}$. By Theorem 2.1, g is L$^\natural$-convex if it satisfies one of the equivalent conditions, (a) to (f), in Theorem 2.2. In particular, L$^\natural$-convex function is a synonym of submodular integrally convex function.

Theorem 2.1, Theorem 3.1 and a general lemma below imply the following.

Theorem 3.2 *The class of L$^\natural$-convex functions and that of M$^\natural$-convex functions are in one-to-one correspondence under the integral Fenchel–Legendre transformation* (3.1). $\Box$

Lemma 3.3 The relation (3.2) between f and $\tilde{f}$ implies (1.4) for $g = f^\bullet$ and $\tilde{g} = \tilde{f}^\bullet$. Conversely, (1.4) between g and $\tilde{g}$ implies (3.2) for $f = g^\bullet$ and $\tilde{f} = \tilde{g}^\bullet$.

(Proof)

$$\begin{aligned} \tilde{f}^\bullet(p_0, p) &= \sup\{p_0 x_0 + \langle p, x \rangle - \tilde{f}(x_0, x) \mid x_0 \in \mathbf{Z}, x \in \mathbf{Z}^V\} \\ &= \sup\{-p_0 x(V) + \langle p, x \rangle - f(x) \mid x \in \mathbf{Z}^V\} \\ &= \sup\{\langle p - p_0 \mathbf{1}, x \rangle - f(x) \mid x \in \mathbf{Z}^V\} \\ &= f^\bullet(p - p_0 \mathbf{1}). \end{aligned}$$

Conversely,

$$
\begin{aligned}
\tilde{g}^\bullet(x_0, x) &= \sup\{p_0 x_0 + \langle p, x\rangle - \tilde{g}(p_0, p) \mid p_0 \in \mathbf{Z}, p \in \mathbf{Z}^V\} \\
&= \sup\{p_0(x_0 + x(V)) + \langle p - p_0\mathbf{1}, x\rangle - g(p - p_0\mathbf{1}) \mid p_0 \in \mathbf{Z}, p \in \mathbf{Z}^V\} \\
&= \sup\{p_0(x_0 + x(V)) + \langle p', x\rangle - g(p') \mid p_0 \in \mathbf{Z}, p' \in \mathbf{Z}^V\} \\
&= \begin{cases} g^\bullet(x) & (x_0 = -x(V)) \\ +\infty & (\text{otherwise}) \end{cases}
\end{aligned}
$$

□

3.2 Duality

A discrete separation theorem is known to hold for L-convex functions. The present result implies that this is also true for L$^\natural$-convex functions.

Similarly to (3.1), define the concave conjugate of a function $\tilde{h} : \mathbf{Z}^{\tilde{V}} \to \mathbf{Z} \cup \{-\infty\}$ by

$$\tilde{h}^\circ(\tilde{p}) = \inf\{\langle \tilde{p}, \tilde{x}\rangle - \tilde{h}(\tilde{x}) \mid \tilde{x} \in \mathbf{Z}^{\tilde{V}}\}, \qquad \tilde{p} \in \mathbf{Z}^{\tilde{V}}. \tag{3.3}$$

We say $\tilde{h}$ is L-concave if $-\tilde{h}$ is L-convex.

The following theorem, called L-separation theorem in [17], may be viewed as a generalization of the discrete separation theorem of Frank [5] for integral submodular set functions.

Theorem 3.4 (Murota [17]) *Let $\tilde{g} : \mathbf{Z}^{\tilde{V}} \to \mathbf{Z} \cup \{+\infty\}$ be an L-convex function and $\tilde{h} : \mathbf{Z}^{\tilde{V}} \to \mathbf{Z} \cup \{-\infty\}$ be an L-concave function such that $\mathrm{dom}\,\tilde{g} \cap \mathrm{dom}\,\tilde{h} \neq \emptyset$ or $\mathrm{dom}\,\tilde{g}^\bullet \cap \mathrm{dom}\,\tilde{h}^\circ \neq \emptyset$. If $\tilde{g}(\tilde{p}) \geq \tilde{h}(\tilde{p})$ ($\tilde{p} \in \mathbf{Z}^{\tilde{V}}$), there exist $\beta \in \mathbf{Z}$ and $\tilde{x} \in \mathbf{Z}^{\tilde{V}}$ such that*

$$\tilde{g}(\tilde{p}) \geq \beta + \langle \tilde{p}, \tilde{x}\rangle \geq \tilde{h}(\tilde{p}), \qquad \tilde{p} \in \mathbf{Z}^{\tilde{V}}.$$

□

As an immediate corollary, we obtain a discrete separation theorem for L$^\natural$-convex functions. Naturally, h is L$^\natural$-concave if $-h$ is L$^\natural$-convex.

Theorem 3.5 *Let $g : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ be an L$^\natural$-convex function and $h : \mathbf{Z}^V \to \mathbf{Z} \cup \{-\infty\}$ be an L$^\natural$-concave function such that $\mathrm{dom}\,g \cap \mathrm{dom}\,h \neq \emptyset$ or $\mathrm{dom}\,g^\bullet \cap \mathrm{dom}\,h^\circ \neq \emptyset$. If $g(p) \geq h(p)$ ($p \in \mathbf{Z}^V$), there exist $\beta \in \mathbf{Z}$ and $x \in \mathbf{Z}^V$ such that*

$$g(p) \geq \beta + \langle p, x\rangle \geq h(p), \qquad p \in \mathbf{Z}^V.$$

(Proof) Define $\tilde{g}(p_0, p_1, \ldots, p_n) = g(p_1 - p_0, \ldots, p_n - p_0)$ and $\tilde{h}(p_0, p_1, \ldots, p_n) = h(p_1 - p_0, \ldots, p_n - p_0)$. The assertion follows from Theorem 3.4 and the following:

$$
\begin{aligned}
& g(p) \geq h(p) \quad (\forall p \in \mathbf{Z}^V) \\
& \Longrightarrow \tilde{g}(p_0, p) \geq \tilde{h}(p_0, p) \quad (\forall (p_0, p) \in \mathbf{Z}^{\tilde{V}}) \\
& \Longrightarrow \exists \beta \in \mathbf{Z}, x_0 \in \mathbf{Z}, x \in \mathbf{Z}^V : \\
& \qquad \tilde{g}(p_0, p) \geq \beta + p_0 x_0 + \langle p, x \rangle \geq \tilde{h}(p_0, p) \quad (\forall (p_0, p) \in \mathbf{Z}^{\tilde{V}}) \\
& \Longrightarrow g(p) \geq \beta + \langle p, x \rangle \geq h(p) \quad (\forall p \in \mathbf{Z}^V).
\end{aligned}
$$

□

The separation theorem above can be reformulated as a Fenchel-type min-max duality relation as follows.

Theorem 3.6 *Let $g : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ be an $L^\natural$-convex function and $h : \mathbf{Z}^V \to \mathbf{Z} \cup \{-\infty\}$ be an $L^\natural$-concave function such that $\mathrm{dom}\, g \cap \mathrm{dom}\, h \neq \emptyset$ or $\mathrm{dom}\, g^\bullet \cap \mathrm{dom}\, h^\circ \neq \emptyset$. Then we have*

$$\inf\{g(p) - h(p) \mid p \in \mathbf{Z}^V\} = \sup\{h^\circ(x) - g^\bullet(x) \mid x \in \mathbf{Z}^V\}.$$

If this common value is finite, the infimum is attained by some $p \in \mathrm{dom}\, g \cap \mathrm{dom}\, h$ and the supremum is attained by some $x \in \mathrm{dom}\, g^\bullet \cap \mathrm{dom}\, h^\circ$. □

See Murota [16, 17] for the Fenchel-type duality on M-/L-convex and concave functions, as well as for the equivalence between the separation theorem and the Fenchel-type duality. It is mentioned that Fujishige [8] formulated the matroid intersection theorem of Edmonds [3] as a Fenchel-type duality for submodular and supermodular functions, whereas Murota [15] rewrote the valuated matroid intersection theorem of Murota [14] as another Fenchel-type duality.

4 Proofs

Proof of Theorem 1.3: The submodularity is obvious from the definition. By submodularity the piecewise-convex extension is given by the Lovász extension in each unit hypercube. By translation property (1.2), the convexity in the neighborhood of the boundary of unit hypercubes is reduced to that in the interior of a unit hypercube. See the proof of Theorem 4.18 of [17] for more details. $\square$

As an immediate corollary of Theorem 1.3 we obtain the following.

Lemma 4.1 *If* $g(p_1, \ldots, p_n) = \tilde{g}(0, p_1, \ldots, p_n)$ *for an L-convex function* $\tilde{g}$*, then* g *satisfies* IC *(integral convexity) and* SBM (2.1). $\square$

Proof of Theorem 2.2: Assume $\tilde{g}(p_0, p_1, \ldots, p_n) = g(p_1 - p_0, \ldots, p_n - p_0)$, which means $\tilde{g}$ satisfies (1.3). Theorem 2.2 is proven according to the following diagram on the basis of a series of lemmas to be established below.

$$\begin{array}{lcl}
\text{(f) SBM \& UPR-SBM} & \underset{\text{(trivial)}}{\Longleftarrow} & \text{(e) PR-SBM} \\
\Downarrow \text{ Lemma 4.6} & & \Updownarrow \text{ Lemma 4.3} \\
\text{(b) MPC} \underset{\text{Lemma 4.7}}{\Longleftarrow} & \text{(a) IC \& SBM} \underset{\text{Lemma 4.1}}{\Longleftarrow} & \text{Submodularity (1.1) of } \tilde{g} \\
\Downarrow \text{ Lemma 4.9} & & \Uparrow \\
\text{(c) L-MPC \& DOM} & & \quad \text{Lemma 4.2} \\
\Downarrow \text{ Lemma 4.10} & & \Downarrow \\
\text{(d) L-SBM} & \Longleftrightarrow & \text{Local submodularity (4.2)} \\
\quad \text{\& L-PR-SBM} & \text{Lemmas 4.4, 4.5} & \text{\& domain condition (4.1)} \\
\quad \text{\& DOM} & & \text{of } \tilde{g}
\end{array}$$

Proof of Theorem 2.1: The "if" part is already shown in Lemma 4.1. The "only if" part can be seen easily from the above diagram. Given a submodular integrally convex function g, define $\tilde{g}$ by $\tilde{g}(p_0, p_1, \ldots, p_n) = g(p_1 - p_0, \ldots, p_n - p_0)$. The second condition (1.3) for L-convexity is met, whereas the above diagram shows that the submodularity (1.1) is implied by (a) IC & SBM. $\square$

A series of lemmas follows.

Lemma 4.2 *Suppose the effective domain of* $\tilde{g} : \mathbf{Z}^{\tilde{n}} \to \mathbf{Z} \cup \{+\infty\}$ *has the property:*

$$\tilde{p}, \tilde{q} \in \mathrm{dom}\, \tilde{g} \quad \Longrightarrow \quad \tilde{p} \vee \tilde{q},\ \tilde{p} \wedge \tilde{q},\ \tilde{p} + \tilde{\mathbf{1}} \in \mathrm{dom}\, \tilde{g}. \tag{4.1}$$

Then submodularity (1.1) *is equivalent to local submodularity:*

$$\tilde{g}(\tilde{p}) + \tilde{g}(\tilde{q}) \geq \tilde{g}(\tilde{p} \vee \tilde{q}) + \tilde{g}(\tilde{p} \wedge \tilde{q}), \qquad ||\tilde{p} - \tilde{q}||_\infty = 1,\ \tilde{p}, \tilde{q} \in \mathbf{Z}^{\tilde{n}}. \tag{4.2}$$

(Proof) We prove (1.1) by induction on $||\tilde{p} - \tilde{q}||_\infty$. Suppose $\tilde{p}, \tilde{q} \in \mathrm{dom}\, \tilde{g}$ and $||\tilde{p} - \tilde{q}||_\infty \geq 2$. For $\tilde{p}^* = ((\tilde{p} \wedge \tilde{q}) + \mathbf{1}) \wedge (\tilde{p} \vee \tilde{q})$ we have $\tilde{p} \wedge \tilde{q} \leq \tilde{p}^* \leq \tilde{p} \vee \tilde{q}$ and $\tilde{p}^* \in \mathrm{dom}\, \tilde{g}$ by (4.1). Since

$$||\tilde{p}^* - \tilde{p}||_\infty \leq ||\tilde{p} - \tilde{q}||_\infty - 1, \quad ||\tilde{p}^* - \tilde{q}||_\infty \leq ||\tilde{p} - \tilde{q}||_\infty - 1,$$

we see by the induction hypothesis that

$$\tilde{g}(\tilde{p}) + \tilde{g}(\tilde{p}^*) \geq \tilde{g}(\tilde{p} \vee \tilde{p}^*) + \tilde{g}(\tilde{p} \wedge \tilde{p}^*), \tag{4.3}$$

$$\tilde{g}(\tilde{q}) + \tilde{g}(\tilde{p}^*) \geq \tilde{g}(\tilde{q} \vee \tilde{p}^*) + \tilde{g}(\tilde{q} \wedge \tilde{p}^*). \tag{4.4}$$

In particular we have $\tilde{p} \vee \tilde{p}^*,\ \tilde{q} \vee \tilde{p}^*,\ \tilde{p} \wedge \tilde{p}^*,\ \tilde{q} \wedge \tilde{p}^*\ \in \mathrm{dom}\, \tilde{g}$. Since

$$||(\tilde{p} \vee \tilde{p}^*) - (\tilde{q} \vee \tilde{p}^*)||_\infty \leq ||\tilde{p} - \tilde{q}||_\infty - 1, \quad ||(\tilde{p} \wedge \tilde{p}^*) - (\tilde{q} \wedge \tilde{p}^*)||_\infty \leq 1,$$

we see again by the induction hypothesis that

$$\begin{aligned}
\tilde{g}(\tilde{p} \vee \tilde{p}^*) + \tilde{g}(\tilde{q} \vee \tilde{p}^*) &\geq \tilde{g}((\tilde{p} \vee \tilde{p}^*) \vee (\tilde{q} \vee \tilde{p}^*)) + \tilde{g}((\tilde{p} \vee \tilde{p}^*) \wedge (\tilde{q} \vee \tilde{p}^*)) \\
&= \tilde{g}(\tilde{p} \vee \tilde{q}) + \tilde{g}(\tilde{p}^*), \qquad (4.5) \\
\tilde{g}(\tilde{p} \wedge \tilde{p}^*) + \tilde{g}(\tilde{q} \wedge \tilde{p}^*) &\geq \tilde{g}((\tilde{p} \wedge \tilde{p}^*) \vee (\tilde{q} \wedge \tilde{p}^*)) + \tilde{g}((\tilde{p} \wedge \tilde{p}^*) \wedge (\tilde{q} \wedge \tilde{p}^*)). \\
&= \tilde{g}(\tilde{p}^*) + \tilde{g}(\tilde{p} \wedge \tilde{q}). \qquad (4.6)
\end{aligned}$$

All the terms in (4.3)∼(4.6) being finite, addition of these inequalities yields (1.1). □

Lemma 4.3 *Assume* $\tilde{g}(p_0, p_1, \ldots, p_n) = g(p_1 - p_0, \ldots, p_n - p_0)$. *Then* $\tilde{g}$ *is L-convex if and only if* g *satisfies* PR-SBM (2.7).

(Proof) The second condition (1.3) for L-convexity of $\tilde{g}$ is automatically met. The submodularity (1.1) of $\tilde{g}$ is translated to a condition on g:

$$g(p - p_0\mathbf{1}) + g(q - q_0\mathbf{1}) \geq g((p \vee q) - (p_0 \vee q_0)\mathbf{1}) + g((p \wedge q) - (p_0 \wedge q_0)\mathbf{1}).$$

We may assume $p_0 \leq q_0$. Change the variables: $p - p_0\mathbf{1} \to p$, $q - q_0\mathbf{1} \to q$, and put $\alpha = q_0 - p_0$. □

This implies the following relationship between $\mathrm{dom}\, \tilde{g}$ and $\mathrm{dom}\, g$.

Lemma 4.4 *Assume* $\tilde{g}(p_0, p_1, \ldots, p_n) = g(p_1 - p_0, \ldots, p_n - p_0)$. *Then* $\mathrm{dom}\, \tilde{g}$ *satisfies* (4.1) *if and only if* $\mathrm{dom}\, g$ *satisfies* DOM (2.4). □

Lemma 4.5 *Assume* $\tilde{g}(p_0, p_1, \ldots, p_n) = g(p_1 - p_0, \ldots, p_n - p_0)$. *Then* $\tilde{g}$ *satisfies the local submodularity* (4.2) *if and only if* g *satisfies* L-SBM (2.5) *and* L-PR-SBM (2.6).

(Proof) The local submodularity (4.2) of $\tilde{g}$ can also be expressed as

$$\tilde{g}(\tilde{p}+\chi_{\tilde{X}})+\tilde{g}(\tilde{p}+\chi_{\tilde{Y}}) \geq \tilde{g}(\tilde{p}+\chi_{\tilde{X}\cup\tilde{Y}})+\tilde{g}(\tilde{p}+\chi_{\tilde{X}\cap\tilde{Y}}), \quad \tilde{X},\tilde{Y} \subseteq \tilde{V},\ \tilde{p} \in \mathbf{Z}^{\tilde{V}}, \tag{4.7}$$

where $\tilde{V} = \{0, 1, \ldots, n\}$. The assertion follows from the relation

$$\tilde{g}(\tilde{p}+\chi_{\tilde{X}}) = \begin{cases} g(p - p_0\mathbf{1} + \chi_X) & (0 \notin \tilde{X}) \\ g(p - (p_0+1)\mathbf{1} + \chi_X) & (0 \in \tilde{X}) \end{cases}$$

where $X = \tilde{X} \setminus \{0\}$, and from a change of variable $p - p_0\mathbf{1} \to p$ or $p - (p_0+1)\mathbf{1} \to p$.

□

Lemma 4.6 SBM (2.1) *&* UPR-SBM (2.8) $\Longrightarrow$ MPC (2.2).

(Proof) Define a sequence of pairs of points $(p^{(k)}, q^{(k)})$ $(k = 0, 1, \cdots)$ as follows:

$$\begin{aligned} &p^{(0)} = p \vee q, \quad q^{(0)} = p \wedge q; \\ &p^{(k+1)} = (p^{(k)} - \mathbf{1}) \vee q^{(k)}, \quad q^{(k+1)} = p^{(k)} \wedge (q^{(k)} + \mathbf{1}) \qquad (k = 0, 1, \cdots). \end{aligned}$$

Note that $p^{(k)} + q^{(k)} = p + q$ for all $k = 0, 1, \cdots$, and that $||p^{(k)} - q^{(k)}||_\infty \leq 1$ if $k \geq ||p - q||_\infty/2$. For N satisfying $N \geq ||p - q||_\infty/2$ we have

$$p^{(N)} \vee q^{(N)} = \left\lceil \frac{p+q}{2} \right\rceil, \qquad p^{(N)} \wedge q^{(N)} = \left\lfloor \frac{p+q}{2} \right\rfloor.$$

Then MPC (2.2) follows from

$$\begin{aligned} g(p) + g(q) &\geq g(p \vee q) + g(p \wedge q) = g(p^{(0)}) + g(q^{(0)}), \\ g(p^{(k)}) + g(q^{(k)}) &\geq g((p^{(k)} - \mathbf{1}) \vee q^{(k)}) + g(p^{(k)} \wedge (q^{(k)} + \mathbf{1})) \\ &= g(p^{(k+1)}) + g(q^{(k+1)}) \qquad (k = 0, 1, \cdots, N-1), \\ g(p^{(N)}) + g(q^{(N)}) &\geq g(p^{(N)} \vee q^{(N)}) + g(p^{(N)} \wedge q^{(N)}) \\ &= g\left(\left\lceil \frac{p+q}{2} \right\rceil\right) + g\left(\left\lfloor \frac{p+q}{2} \right\rfloor\right). \end{aligned}$$

□

Lemma 4.7 IC *(integral convexity) &* SBM (2.1) $\Longrightarrow$ MPC (2.2).

(Proof) This implication is essentially proved by Favati–Tardella [4]. We describe the proof for completeness. Since g is submodular, the piecewise-convex extension $\overline{g}$ is given by the Lovász extension in each unit hypercube. Therefore,

$$2\overline{g}\left(\frac{p+q}{2}\right) = g\left(\left\lceil\frac{p+q}{2}\right\rceil\right) + g\left(\left\lfloor\frac{p+q}{2}\right\rfloor\right).$$

The convexity of $\overline{g}$, on the other hand, implies

$$g(p)+g(q) = \overline{g}(p)+\overline{g}(q) \geq 2\overline{g}\left(\frac{p+q}{2}\right).$$

A combination of these two yields MPC (2.2). $\square$

Lemma 4.8 DOM (2.4) *follows from the condition:*

$$p, q \in \mathrm{dom}\, g \quad \Longrightarrow \quad \left\lceil\frac{p+q}{2}\right\rceil, \ \left\lfloor\frac{p+q}{2}\right\rfloor \ \in \mathrm{dom}\, g. \tag{4.8}$$

(Proof) For any $p, q \in \mathrm{dom}\, g$ define a sequence $(q^{(0)}, q^{(1)}, \cdots)$ of points in $\mathbf{Z}^n$ as follows:

$$q^{(0)} = q; \qquad q^{(k+1)} = \left\lfloor\frac{p+q^{(k)}}{2}\right\rfloor \quad (k = 0, 1, \cdots).$$

Here, note that $q^{(k)} \in \mathrm{dom}\, g$ $(k = 0, 1, \cdots)$. We see that

(i) if $p_i - q_i^{(k)} = 0$ or 1, then $q_i^{(k+1)} = q_i^{(k)}$;

(ii) if $p_i - q_i^{(k)} \geq 2$, then $p_i > q_i^{(k+1)} = q_i^{(k)} + \lfloor\frac{1}{2}(p_i - q_i^{(k)})\rfloor \geq q_i^{(k)} + 1$;

(iii) if $p_i - q_i^{(k)} \leq -1$, then $p_i \leq q_i^{(k+1)} = q_i^{(k)} - \lceil\frac{1}{2}(q_i^{(k)} - p_i)\rceil \leq q_i^{(k)} - 1$.

It follows that there exists some positive integer N such that $q^{(k)} = q^{(N)}$ for any integer $k \geq N$. Because of (i)∼(iii) such $q^{(N)}$ is equal to $(p-\mathbf{1}) \vee (p \wedge q)$ and hence we have $(p-\mathbf{1}) \vee (p \wedge q) \in \mathrm{dom}\, g$. Replacing p by $(p-\mathbf{1}) \vee (p \wedge q)$ and repeating the above argument, we also have $(p - 2\cdot\mathbf{1}) \vee (p \wedge q) \in \mathrm{dom}\, g$. Repeating this argument (or more rigorously by induction), we have $(p - \alpha\mathbf{1}) \vee (p \wedge q) \in \mathrm{dom}\, g$ for $0 \leq \alpha \in \mathbf{Z}$. In particular, we have $p \wedge q \in \mathrm{dom}\, g$. By the symmetry we also have $(p \vee q) \wedge (q + \alpha\mathbf{1}) \in \mathrm{dom}\, g$ for $0 \leq \alpha \in \mathbf{Z}$ and, in particular, $p \vee q \in \mathrm{dom}\, g$.

Now, replacing p by $p \vee q$ in the above argument from the beginning, we have $(p - \alpha\mathbf{1}) \vee q \in \mathrm{dom}\, g$ for $0 \leq \alpha \in \mathbf{Z}$. By the symmetry we also have $p \wedge (q + \alpha\mathbf{1}) \in \mathrm{dom}\, g$ for $0 \leq \alpha \in \mathbf{Z}$. $\square$

Lemma 4.9 MPC (2.2) $\Longrightarrow$ L-MPC (2.3) *&* DOM (2.4).

(Proof) Obviously, MPC (2.2) $\Longrightarrow$ L-MPC (2.3). MPC (2.2) implies (4.8), which in turn implies DOM (2.4) by Lemma 4.8. $\square$

Lemma 4.10 L-MPC (2.3) $\Longrightarrow$ L-SBM (2.5) *&* L-PR-SBM (2.6).

(Proof) L-MPC (2.3) for $p = \hat{p} + \chi_X$ and $q = \hat{p} + \chi_Y$ implies (2.5), since

$$\left\lceil \frac{p+q}{2} \right\rceil = \hat{p} + \chi_{X \cup Y}, \quad \left\lfloor \frac{p+q}{2} \right\rfloor = \hat{p} + \chi_{X \cap Y},$$

and $||p - q||_\infty \leq 1$, whereas L-MPC (2.3) for $p = \hat{p} + \chi_X + \mathbf{1}$ and $q = \hat{p} + \chi_Y$ implies (2.6), since

$$\left\lceil \frac{p+q}{2} \right\rceil = \hat{p} + \chi_{X \cap Y} + \mathbf{1}, \quad \left\lfloor \frac{p+q}{2} \right\rfloor = \hat{p} + \chi_{X \cup Y}$$

and $||p - q||_\infty \leq 2$. $\square$

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