Short Proofs of the Separation Theorems for L-convex/concave and M-convex/concave Functions

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Abstract

Recently K. Murota has introduced concepts of L-convex function and M-convex function as generalizations of those of submodular function and base polyhedron, respectively, and has shown separation theorems for L-convex/concave functions and for M-convex/concave functions. The present note gives short alternative proofs of the separation theorems by relating them to the ordinary separation theorem in convex analysis with the full use of recently obtained knowledge about L-/M-convex functions.

Keywords: Separation Theorems, L-convex Functions, M-convex Functions

1. Introduction

Recently Murota [13], [14], [15] has introduced concepts of L-convex function and M-convex function as generalizations of those of submodular function and base polyhedron, respectively, and has shown separation theorems for L-convex/concave functions and for M-convex/concave functions, named the L-separation theorem and the M-separation theorem, respectively (also see [1], [2], [12], [11], [18], [19] for related works). The equivalent variants of L-convex and M-convex functions, called L*-convex and M*-convex functions, are considered by Fujishige and Murota [8] and...
by Murota and Shioura [16], respectively. The L- and M-separation theorems can equivalently be reformulated for $L^3$-convex and $M^3$-convex functions.

The present note is to give short alternative proofs of the separation theorems formulated for $L^3$-convex/concave and $M^3$-convex/concave functions. The proofs will be given by relating the theorems to the ordinary separation theorem in convex analysis ([17]) with the full use of recently obtained knowledge about $L^3$-convex and $M^3$-convex functions, especially the conjugacy relation between $L^3$-convex and $M^3$-convex functions. In contrast, the original proofs are based on algorithmic (constructive) arguments.

2. Definitions and Preliminaries

Let $V$ be a nonempty finite set and let $Z$ and $R$, respectively, be the sets of integers and of reals. Also let $v_0$ be a new element not in $V$ and define $\tilde{V} = \{v_0\} \cup V$. For any function $f : Z^V \to Z \cup \{+\infty\}$ (or $g : Z^V \to Z \cup \{-\infty\}$) we define $\text{dom} f = \{p \in Z^V \mid f(p) < +\infty\}$ (or $\text{dom} g = \{p \in Z^V \mid g(p) > -\infty\}$).

A function $f : Z^V \to Z \cup \{+\infty\}$ with $\text{dom} f \neq \emptyset$ is called $L$-convex if

$$f(p) + f(q) \geq f(p \lor q) + f(p \land q) \quad (p, q \in Z^V), \quad (2.1)$$

$$\exists r \in Z : f(p + 1) = f(p) + r \quad (p \in Z^V), \quad (2.2)$$

where $p \lor q = (\max\{p(u), q(u)\} \mid u \in V)$, $p \land q = (\min\{p(u), q(u)\} \mid u \in V)$ and $1$ is the vector in $Z^V$ with all of its components being equal to 1. A function $f : Z^V \to Z \cup \{+\infty\}$ with $\text{dom} f \neq \emptyset$ is called $L^3$-convex if it is expressed in terms of an $L$-convex function $\tilde{f} : \tilde{Z}^V \to \tilde{Z} \cup \{+\infty\}$ as

$$f(p) = \tilde{f}(0, p). \quad (2.3)$$

It has been shown in [8] that an $L^3$-convex function is exactly a submodular integrally convex function considered by Favati and Tardella [3] except for a domain condition. Note that $f$ is a restriction of $\tilde{f}$ to $Z^V$ and that $f$ uniquely determines $\tilde{f}$ up to a constant $r$ as in (2.2). A set $P \subseteq Z^V$ is called an $L^3$-convex set if $P = \text{dom} f$ for some $L^3$-convex function (cf. [14]). It follows from [8] that $P \subseteq Z^V$ is an $L^3$-convex set if and only if

$$p, q \in P \quad \implies \quad \left\lfloor \frac{p + q}{2} \right\rfloor, \left\lceil \frac{p + q}{2} \right\rceil \in P, \quad (2.4)$$

where for any $p \in R^V$ we denote by $[p]$ (or $\lfloor p \rfloor$) the vector with each component obtained by rounding up (or down) to the nearest integer the corresponding component of $p$. Hence, the intersection of two (or more than two) $L^3$-convex sets, if nonempty, is also an $L^3$-convex set. We call the convex hull, in $R^V$, of an $L^3$-convex set an integral $L^3$-convex polyhedron.
A function \( \varphi : \mathbb{Z}^V \to \mathbb{Z} \cup \{+\infty\} \) with \( \text{dom}\varphi \neq \emptyset \) is called \( M \)-convex if it satisfies

\[ (\text{M-EXC}) \quad \text{For any } x, y \in \text{dom}\varphi \text{ and any } u \in \text{Supp}^+(x - y) \text{ there exists an element } v \in \text{Supp}^-(x - y) \text{ such that} \]

\[ \varphi(x) + \varphi(y) \geq \varphi(x - \chi_u + \chi_v) + \varphi(y + \chi_u - \chi_v), \quad (2.5) \]

where for any \( z \in \mathbb{Z}^V \), \( \text{Supp}^+(z) = \{ u \in V \mid z(u) > 0 \} \) and \( \text{Supp}^-(z) = \{ u \in V \mid z(u) < 0 \} \) and for any \( W \subseteq V \), \( \chi_W \) is the characteristic vector of \( W \) and \( \chi_u = \chi_{\{u\}} \) for \( u \in V \).

A function \( \varphi : \mathbb{Z}^V \to \mathbb{Z} \cup \{+\infty\} \) with \( \text{dom}\varphi \neq \emptyset \) is called \( M^\ast \)-convex if it is expressed in terms of an \( M \)-convex function \( \tilde{\varphi} : \tilde{\mathbb{Z}}^V \to \mathbb{Z} \cup \{+\infty\} \) as

\[ \tilde{\varphi}(k, x) = \begin{cases} \varphi(x) & \text{if } k + \sum_{u \in V} x(u) = 0 \\ +\infty & \text{otherwise.} \end{cases} \quad (2.6) \]

Note that \( \varphi \) is a projection of \( \tilde{\varphi} \). A set \( Q \subseteq \mathbb{Z}^V \) is called an \( M^\ast \)-convex set if \( Q = \text{dom}\varphi \) for some \( M \)-convex function \( \varphi \) (cf. [14]). It follows from [16] that \( Q \subseteq \mathbb{Z}^V \) is an \( M^\ast \)-convex set if and only if \( Q \) is the set of integer points of an integral generalized polymatroid \( (g\text{-polymatroid}) \). (For \( g \)-polymatroids see [4], [5] and [7].) Hence, we refer to an integral \( g \)-polymatroid as an integral \( M^\ast \)-convex polyhedron.

Most of the important properties of \( L^\ast /M^\ast \)-convex functions established in [13], [14], [15], [19] can be transferred to \( L^2 /M^2 \)-convex functions through the restriction/projection operation defined above.

Before getting into the proofs of the separation theorems we summarize some basic properties of \( L^2 \)- and \( M^2 \)-convex functions (see [13], [14], [15], [16], [8]). We say that a convex polyhedron is integral if each nonempty face of it contains an integer point. For a piecewise-linear convex (or concave) function \( f \) on \( \mathbb{R}^V \) a maximal subset of its domain on which \( f \) is linear is called a linearity domain of \( f \).

1. Any \( L^2 \)-convex function \( f : \mathbb{Z}^V \to \mathbb{Z} \cup \{+\infty\} \) can be extended to a piecewise-linear convex function \( \tilde{f} : \mathbb{R}^V \to \mathbb{R} \cup \{+\infty\} \) with \( \tilde{f}(p) = f(p) \) for all \( p \in \mathbb{Z}^V \), where for each \( p \in \mathbb{R}^V \)

\[ \tilde{f}(p) = \sup\{ f_0(p) \mid f_0 \text{ is a convex function on } \mathbb{R}^V, \forall q \in \mathbb{Z}^V : f_0(q) \leq f(q) \}. \]

Roughly speaking, the piecewise-linear convex extension \( \tilde{f} \) is obtained by pasting a collection of translated truncated Lovász extensions of submodular set functions (cf. [3]). (For the truncated Lovász extension of submodular functions see [6] and [9].)

2. Any \( M^2 \)-convex function \( \varphi : \mathbb{Z}^V \to \mathbb{Z} \cup \{+\infty\} \) can be extended to a piecewise-linear convex function \( \tilde{\varphi} : \mathbb{R}^V \to \mathbb{R} \cup \{+\infty\} \) with \( \tilde{\varphi}(x) = \varphi(x) \) for all \( x \in \mathbb{Z}^V \).
Since \( f \) (resp. \( \varphi \)) uniquely determines \( \bar{f} \) (resp. \( \bar{\varphi} \)) and vice versa, we identify \( f \) (resp. \( \varphi \)) with \( \bar{f} \) (resp. \( \bar{\varphi} \)). For example, we denote by \( \partial f(p) \subseteq \mathbb{R}^V \) for \( p \in \mathbb{Z}^V \) the subdifferential of \( f \) at \( p \) in the ordinary sense of convex analysis ([17]). When we consider an \( L^2 \)-concave function \( g \), we denote by \( \partial g(p) \) the “superdifferential” of \( g \) at \( p \), i.e., \( \partial g(p) = -\partial(-g)(p) \). We adopt similar notations for \( M^2 \)-convex/concave functions.

For an \( L^3 \)-convex (or \( L^2 \)-concave) function \( f \) (or \( g \)) we denote by \( f^* \) (or \( g^* \)) the convex (or concave) conjugate of \( f \) (or \( g \)), i.e.,

\[
f^*(x) = \sup\{ \langle p, x \rangle - f(p) \mid p \in \mathbb{Z}^V \} \quad (\text{or} \quad g^*(x) = \inf\{ \langle p, x \rangle - g(p) \mid p \in \mathbb{Z}^V \}),
\]

where \( \langle \cdot, \cdot \rangle \) denotes the canonical inner product (duality pairing). Similarly, we denote by \( \varphi^* \) (or \( \psi^* \)) the convex (or concave) conjugate of an \( M^2 \)-convex (or \( M^2 \)-concave) function \( \varphi \) (or \( \psi \)), i.e.,

\[
\varphi^*(p) = \sup\{ \langle p, x \rangle - \varphi(x) \mid x \in \mathbb{Z}^V \} \quad (\text{or} \quad \psi^*(p) = \inf\{ \langle p, x \rangle - \psi(x) \mid x \in \mathbb{Z}^V \}).
\]

3. An \( L^2 \)-convex function is the convex conjugate of an \( M^2 \)-convex function and vice versa.

4. For the piecewise-linear convex extension \( \bar{f} \) of an \( L^2 \)-convex function \( f : \mathbb{Z}^V \to \mathbb{Z} \cup \{+\infty\} \), each linearity domain of \( \bar{f} \) is an integral \( L^3 \)-convex polyhedron.

5. For each integer point \( p \in \mathbb{Z}^V \) the subdifferential \( \partial f(p) \) of an \( L^2 \)-convex function \( f : \mathbb{Z}^V \to \mathbb{Z} \cup \{+\infty\} \) at \( p \) is an integral \( g \)-polymatroid.

6. For the piecewise-linear convex extension \( \bar{\varphi} \) of an \( M^2 \)-convex function \( \varphi : \mathbb{Z}^V \to \mathbb{Z} \cup \{+\infty\} \), each linearity domain of \( \bar{\varphi} \) is an integral \( g \)-polymatroid, an \( M^2 \)-convex polyhedron.

7. For each integer point \( x \in \mathbb{Z}^V \) the subdifferential \( \partial \varphi(x) \) of an \( M^2 \)-convex function \( \varphi : \mathbb{Z}^V \to \mathbb{Z} \cup \{+\infty\} \) at \( x \) is an integral \( L^2 \)-convex polyhedron.

These properties fully characterize (integer-valued) \( L^2 \)-convex and \( M^2 \)-convex functions defined on \( \mathbb{Z}^V \). Also note the following fact.

8. The intersection of two integral \( g \)-polymatroids (\( M^2 \)-convex polyhedra), if nonempty, is an integral polyhedron (see [5]) and that the intersection of two integral \( L^2 \)-convex polyhedra, if nonempty, is again an integral \( L^2 \)-convex polyhedron (see [14]).

We utilize these properties in the following argument.
3. The Separation Theorems

We now give a short proof of the following theorem, the L-separation theorem formulated for \( L^e \)-convex and \( L^c \)-concave functions.

**Theorem 3.1** (Murota [14]): Let \( f : \mathbb{Z}^V \to \mathbb{Z} \cup \{+\infty\} \) and \( g : \mathbb{Z}^V \to \mathbb{Z} \cup \{-\infty\} \), respectively, be \( L^e \)-convex and \( L^c \)-concave functions such that \( \text{dom} f \cap \text{dom} g \neq \emptyset \) or \( \text{dom} f^* \cap \text{dom} g^0 \neq \emptyset \). If we have

\[
f(p) \geq g(p) \quad (p \in \mathbb{Z}^V),
\]

then there exist an integral vector \( x \in \mathbb{Z}^V \) and an integer \( \alpha \in \mathbb{Z} \) such that

\[
f(p) \geq \langle p, x \rangle + \alpha \geq g(p) \quad (p \in \mathbb{Z}^V).
\]

(Proof) Suppose that \( \text{dom} f \cap \text{dom} g \neq \emptyset \). Then, define

\[
k = \min \{ f(p) - g(p) \mid p \in \mathbb{Z}^V \}
\]

and let \( p_0 \) be a vector \( p \) that attains the minimum value \( k \) in (3.3). It suffices to prove the present theorem in the case when \( k = 0 \). Then, since \( f(p_0) = g(p_0) \), a vector \( x \in \mathbb{R}^V \) satisfies (3.2) for some \( \alpha \) if and only if

\[x \in \partial f(p_0) \cap \partial g(p_0).\] (3.4)

On the other hand, both \( \partial f(p_0) \) and \( \partial g(p_0) \) are integral \( g \)-polymatroids and we have \( \partial f(p_0) \cap \partial g(p_0) \neq \emptyset \) in \( \mathbb{R}^V \) due to the ordinary separation theorem in convex analysis. It follows from the integrality property of the intersection of two integral \( g \)-polymatroids that there exists an integral vector \( x \in \mathbb{Z}^V \) that satisfies (3.4) and hence (3.2) with \( \alpha = g(p_0) - \langle p_0, x \rangle \in \mathbb{Z} \).

In the case when \( \text{dom} f \cap \text{dom} g = \emptyset \) and \( \text{dom} f^* \cap \text{dom} g^0 \neq \emptyset \), take \( x_0 \in \text{dom} f^* \cap \text{dom} g^0 \). From the inequalities

\[
f^*(x) = \sup_{p \in \text{dom} f} \{ \langle p, x - x_0 \rangle + [\langle p, x_0 \rangle - f(p)] \} \leq \sup_{p \in \text{dom} f} \langle p, x - x_0 \rangle + f^*(x_0),
\]

\[
g^0(x) = \inf_{p \in \text{dom} g} \{ \langle p, x - x_0 \rangle + [\langle p, x_0 \rangle - g(p)] \} \geq \inf_{p \in \text{dom} g} \langle p, x - x_0 \rangle + g^0(x_0),
\]

we obtain

\[
g^0(x) - f^*(x) \geq \left[ \inf_{p \in \text{dom} g} \langle p, x - x_0 \rangle - \sup_{p \in \text{dom} f} \langle p, x - x_0 \rangle \right] + g^0(x_0) - f^*(x_0). \] (3.5)

Since \( \text{dom} f \cap \text{dom} g = \emptyset \), the convex hulls of \( \text{dom} g \) and \( \text{dom} f \) are disjoint by the integrality property of the intersection of \( L^e \)-convex polyhedra. It then follows from
the (rational) separation theorem stated in Lemma 3.2 below that there exists an integral vector \( x = x^* \in \mathbb{Z}^V \) that makes the right-hand side of (3.5) large enough to guarantee \( g^0(x^*) - f^*(x^*) > 0. \) (Note that the convex hulls of \( \text{dom}g \) and \( \text{dom}f \), each being an \( L^2 \)-convex polyhedron, can be described by a finite number of inequalities with integer coefficients and integer right-hand sides.) This implies (3.2) with \( x = x^* \) and an integer \( \alpha \) such that \( f^*(x^*) \leq -\alpha \leq g^0(x^*) \). \( \square \)

**Lemma 3.2:** Let \( P_1 \) and \( P_2 \) be polyhedra described as \( P_i = \{ p \in \mathbb{R}^V \mid A_ip \leq b_i \} \) \((i = 1, 2)\) with integral matrices \( A_i \) and integral vectors \( b_i \) \((i = 1, 2)\). If \( P_1 \cap P_2 = \emptyset \), then for any large positive integer \( N \) there exists an integral vector \( x^* \) such that

\[
\inf\{\langle p, x^* \rangle \mid p \in P_1 \} - \sup\{\langle p, x^* \rangle \mid p \in P_2 \} \geq N. \tag{3.6}
\]

(Proof) By the ordinary separation theorem for polyhedra there exists a rational vector \( x \) such that

\[
\inf\{\langle p, x \rangle \mid p \in P_1 \} - \sup\{\langle p, x \rangle \mid p \in P_2 \} \geq \epsilon > 0. \tag{3.7}
\]

The desired integral vector is obtained as \( x^* = Mx \) with an integer \( M \geq N/\epsilon \) that is a common multiple of the denominators of the components of \( x \). \( \square \)

Next, we give an alternative short proof of the following theorem, the \( M \)-separation theorem formulated for \( M^2 \)-convex and \( M^2 \)-concave functions. This is the conjugate counterpart of the \( L \)-separation theorem.

**Theorem 3.3** \( \text{[Murota [13], [14]]} \): Let \( \varphi : \mathbb{Z}^V \to \mathbb{Z} \cup \{-\infty\} \) and \( \psi : \mathbb{Z}^V \to \mathbb{Z} \cup \{+\infty\} \), respectively, be \( M^2 \)-convex and \( M^2 \)-concave functions such that \( \text{dom}\varphi \cap \text{dom}\psi = \emptyset \) or \( \text{dom}\varphi^* \cap \text{dom}\psi^* = \emptyset \). If we have

\[
\varphi(x) \geq \psi(x) \quad (x \in \mathbb{Z}^V), \tag{3.8}
\]

then there exist an integral vector \( p \in \mathbb{Z}^V \) and an integer \( \beta \in \mathbb{Z} \) such that

\[
\varphi(x) \geq \langle p, x \rangle + \beta \geq \psi(x) \quad (x \in \mathbb{Z}^V). \tag{3.9}
\]

(Proof) Suppose that \( \text{dom}\varphi \cap \text{dom}\psi \neq \emptyset \). Then, similarly as in the proof of Theorem 3.1, we can assume that for some \( x_0 \in \mathbb{Z}^V \) we have \( \varphi(x_0) = \psi(x_0) \). Since \( \varphi(x_0) = \psi(x_0) \), a vector \( p \in \mathbb{R}^V \) satisfies (3.9) for some \( \beta \) if and only if

\[
p \in \partial\varphi(x_0) \cap \partial\psi(x_0). \tag{3.10}
\]

On the other hand, both \( \partial\varphi(x_0) \) and \( \partial\psi(x_0) \) are integral \( L^2 \)-convex polyhedra and we have \( \partial\varphi(x_0) \cap \partial\psi(x_0) \neq \emptyset \) in \( \mathbb{R}^V \) due to the ordinary separation theorem in convex analysis. Since the intersection \( \partial\varphi(x_0) \cap \partial\psi(x_0) \) is an integral \( L^2 \)-convex polyhedron, there exists an integral vector \( p \in \mathbb{Z}^V \) that satisfies (3.10) and hence (3.9) with \( \beta = \psi(p_0) - \langle p, x_0 \rangle \in \mathbb{Z} \).

The proof for the case of \( \text{dom}\varphi \cap \text{dom}\psi = \emptyset \) and \( \text{dom}\varphi^* \cap \text{dom}\psi^* = \emptyset \) can be reduced to a separation theorem (Lemma 3.2) for \( M^2 \)-convex polyhedra as in the proof of Theorem 3.1. \( \square \)
References


